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DYNAMICS AND CONTROL OF A BIOMIMETIC VEHICLE USING BIASED WINGBEAT FORCING FUNCTIONS: PART II - CONTROLLER (POSTPRINT)

David B. Doman, Michael W. Oppenheimer, and David O. Sigthorsson

Control Design and Analysis Branch
Control Sciences Division

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14. ABSTRACT
A control strategy is proposed for a minimally-actuated flapping-wing micro air-vehicle (FWMAV). The proposed vehicle is similar to the Harvard RoboFly that accomplished the first takeoff of an insect scale flapping wing aircraft, except that it is equipped with independently actuated wings. Using the derivation of the aerodynamic forces and moments from Part I, a control allocation strategy and a feedback control law are designed that enable the vehicle to achieve untethered, stabilized flight about a hover condition. Six degree-of-freedom maneuvers near hover are demonstrated as well. The control laws are designed to make use of two actuators that control the angular position of the wing in the stroke plane. The Split-Cycle Constant-Period Frequency Modulation with Wing Bias technique, introduced in Part I, is used to allow each wing to generate non-zero cycleaveraged aerodynamic forces and moments. This technique modifies the frequencies of the up and down strokes to yield non-zero cycle-averaged drag due to the flapping motion of a wing. Additionally, the midpoint of the wingbeat profile can be modified by use of a wing bias. The bias is introduced to primarily provide pitching moment control. In this work, the sensitivities of cycle-averaged forces and moments with respect to the control parameters are computed. Single axis controllers are designed and the complete system is simulated.

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Dynamics and Control of a Biomimetic Vehicle Using Biased Wingbeat Forcing Functions: Part II - Controller

David B. Doman *

Michael W. Oppenheimer †

David O. Sigthorsson ‡

A control strategy is proposed for a minimally-actuated flapping-wing micro air-vehicle (FWMAV). The proposed vehicle is similar to the Harvard RoboFly that accomplished the first takeoff of an insect scale flapping wing aircraft, except that it is equipped with independently actuated wings. Using the derivation of the aerodynamic forces and moments from Part I, a control allocation strategy and a feedback control law are designed that enable the vehicle to achieve untethered, stabilized flight about a hover condition. Six degree-of-freedom maneuvers near hover are demonstrated as well. The control laws are designed to make use of two actuators that control the angular position of the wing in the stroke plane. The Split-Cycle Constant-Period Frequency Modulation with Wing Bias technique, introduced in Part I, is used to allow each wing to generate non-zero cycle-averaged aerodynamic forces and moments. This technique modifies the frequencies of the up and down strokes to yield non-zero cycle-averaged drag due to the flapping motion of a wing. Additionally, the midpoint of the wingbeat profile can be modified by use of a wing bias. The bias is introduced to primarily provide pitching moment control. In this work, the sensitivities of cycle-averaged forces and moments with respect to the

*Senior Aerospace Engineer, Control Design and Analysis Branch, 2210 Eighth Street, Ste 21, Air Force Research Laboratory, WPAFB, OH 45433-7531 Email David.Doman@wpafb.af.mil, Ph. (937) 255-8451, Fax (937) 656-4000, Associate Fellow AIAA

†Senior Electronics Engineer, Control Design and Analysis Branch, 2210 Eighth Street, Ste 21, Air Force Research Laboratory, WPAFB, OH 45433-7531 Email Michael.Oppenheimer@wpafb.af.mil, Ph. (937) 255-8490, Fax (937) 656-4000, Associate Fellow AIAA

‡Electronics Engineer, Control Design and Analysis Branch, 2210 Eighth Street, Ste 21, Air Force Research Laboratory, WPAFB, OH 45433-7531 Email David.Sigthorsson@afmcx.net, Ph. (937) 255-9707, Fax (937) 656-4000, Member AIAA. This research was performed while this author held a National Research Council Research Associateship Award at the Air Force Research Laboratory.

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control parameters are computed. Single axis controllers are designed and the complete system is simulated.

I. Introduction

In Part I[1] of this series, expressions for the instantaneous and cycle-averaged aerodynamic forces and moments of a flapping wing biomimetic vehicle were developed. This vehicle is similar to the Harvard Robofly described by Wood,[2] with the primary exception being the proposed vehicle makes use of two actuators, which are used to independently vary the position of the wing spars in the stroke plane. In previous work,[3,4] a third actuator was used to control the position of a bobweight such that the center-of-gravity could be manipulated to control the pitching moment. In this work, the third actuator has been eliminated by the inclusion of a wing bias term. The goal of this paper is to evaluate the cycle-averaged control derivatives with respect to all control variables to determine if sufficient control authority can be achieved to regulate the vehicle's 6 degree-of-freedom (DOF) body position and attitude using wingbeat frequencies, ω_{RW} and ω_{LW}, the split-cycle parameters, δ_{RW} and δ_{LW}, and the wing bias terms, η_{RW} and η_{LW}, as control inputs.

In the present paper, the analysis that was started in Part I[1] is continued by analyzing the effects of wing bias and other control parameters upon the body-axis forces and moments. Expressions for the cycle-averaged control derivatives associated with the variations in the fundamental wingbeat frequencies, split-cycle parameters, and wing biases are derived. The expressions are linearized about a hover flight condition and a control allocation strategy is proposed. A controller is developed such that the vehicle can perform waypoint tracking and simulation results are presented.

II. Control Derivatives

The parameters used to control the aerodynamic forces and moments are the fundamental wingbeat frequencies, ω_{RW} and ω_{LW}, the split-cycle parameters, δ_{RW} and δ_{LW}, and the wing bias terms, η_{RW} and η_{LW}. For controllability analysis and control synthesis, the sensitivity of each cycle-averaged force and moment to each control input parameter must be determined. The forces and moments are, in general, functions of the control input parameters as well as the past values of η_{RW}, η_{LW} because when $A_{RW} = A_{LW} = 1$, $\Delta A_i = \eta_i - \eta_{i-1} = \eta - \eta_{-1}$. Note that the subscript i notation has been removed and it should be understood that $\eta = \eta_i$ (the current value of wing bias) and $\eta_{-1} = \eta_{i-1}$ (the one cycle past value of wing bias). Hence, the cycle-averaged forces and moments are sensitive to changes in η_{-1}, although η_{-1} is not considered a control input. Letting $\bar{G}$ be a generalized cycle-averaged force or moment, the

total increment of $\bar{G}$ is

$$\begin{aligned}\Delta\bar{G} = \frac{\partial\bar{G}}{\partial\omega_{RW}}\Delta\omega_{RW} + \frac{\partial\bar{G}}{\partial\omega_{LW}}\Delta\omega_{LW} + \frac{\partial\bar{G}}{\partial\delta_{RW}}\delta_{RW} + \frac{\partial\bar{G}}{\partial\delta_{LW}}\delta_{LW} \\ + \frac{\partial\bar{G}}{\partial\eta_{RW}}\eta_{RW} + \frac{\partial\bar{G}}{\partial\eta_{LW}}\eta_{LW} + \frac{\partial\bar{G}}{\partial\eta_{RW-1}}\eta_{RW-1} + \frac{\partial\bar{G}}{\partial\eta_{LW-1}}\eta_{LW-1}\end{aligned} \tag{1}$$

A feedback controller will generate a set of desired cycle-averaged forces and moments, hence, the left-hand side of Equation 1 will be replaced $\Delta\bar{G}_{des}$. Of interest is the hover condition and Equation 1 can be written as

$$\begin{aligned}\Delta\bar{G}_{des} = \left.\frac{\partial\bar{G}}{\partial\omega_{RW}}\right|_{hover}\Delta\omega_{RW} + \left.\frac{\partial\bar{G}}{\partial\omega_{LW}}\right|_{hover}\Delta\omega_{LW} + \left.\frac{\partial\bar{G}}{\partial\delta_{RW}}\right|_{hover}\delta_{RW} + \left.\frac{\partial\bar{G}}{\partial\delta_{LW}}\right|_{hover}\delta_{LW} \\ + \left.\frac{\partial\bar{G}}{\partial\eta_{RW}}\right|_{hover}\eta_{RW} + \left.\frac{\partial\bar{G}}{\partial\eta_{LW}}\right|_{hover}\eta_{LW} + \left.\frac{\partial\bar{G}}{\partial\eta_{RW-1}}\right|_{hover}\eta_{RW-1} + \left.\frac{\partial\bar{G}}{\partial\eta_{LW-1}}\right|_{hover}\eta_{LW-1}\end{aligned} \tag{2}$$

The increments in some parameters are replaced by the parameters themselves since, at hover, $\delta_{RW} = \delta_{LW} = \eta_{RW} = \eta_{LW} = \eta_{RW-1} = \eta_{LW-1} = 0$. The partial derivatives in Equation 2 are now evaluated. The expressions for the cycle-averaged forces and moments are given in Part I[1] of this series.

A. X-Body Axis Force Control Derivatives

$$\frac{\partial\overline{F}_{xRW}^{B}}{\partial\delta_{RW}} = \frac{k_L}{4}\left\{-A_{RW}^2\omega_{RW} + \frac{\omega_{RW}^3}{2\left(\omega_{RW} - 2\delta_{RW}\right)^2}\left(A_{RW}^2 + \left(A_{RW} + \Delta A_{RW}\right)^2\right)\right\} \tag{3}$$

$$\begin{aligned}\frac{\partial\overline{F}_{xRW}^{B}}{\partial\omega_{RW}} = \frac{k_L}{4}\Bigg\{A_{RW}^2\left(2\omega_{RW} - \delta_{RW}\right) + \left(\frac{2\omega_{RW}^3 - 7\delta_{RW}\omega_{RW}^2 + 4\omega_{RW}\delta_{RW}^2}{2\left(\omega_{RW} - 2\delta_{RW}\right)^2}\right) \\ * \left(A_{RW}^2 + \left(A_{RW} + \Delta A_{RW}\right)^2\right)\Bigg\}\end{aligned} \tag{4}$$

For the x-body axis force sensitivity with respect to η_{RW}, recall that ΔA_{RW} is a function of η_{RW}. Therefore,

$$\frac{\partial\overline{F}_{xRW}^{B}}{\partial\eta_{RW}} = \frac{k_L\omega_{RW}\left(\omega_{RW} + \sigma_{RW}\right)}{4}\left(A_{RW} + \Delta A_{RW}\right)\frac{\partial\Delta A_{RW}}{\partial\eta_{RW}} \tag{5}$$

$$\frac{\partial\overline{F}_{xRW}^{B}}{\partial\eta_{RW-1}} = \frac{k_L\omega_{RW}\left(\omega_{RW} + \sigma_{RW}\right)}{4}\left(A_{RW} + \Delta A_{RW}\right)\frac{\partial\Delta A_{RW}}{\partial\eta_{RW-1}} \tag{6}$$

Left wing expressions are of the same form as the right wing.

$$\frac{\partial \overline{F}_{xLW}^{B}}{\partial \delta_{LW}} = \frac{k_L}{4}\left\{-A_{LW}^2\omega_{LW} + \frac{\omega_{LW}^3}{2\left(\omega_{LW}-2\delta_{LW}\right)^2}\left(A_{LW}^2 + \left(A_{LW}+\Delta A_{LW}\right)^2\right)\right\} \tag{7}$$

$$\begin{aligned}\frac{\partial \overline{F}_{xLW}^{B}}{\partial \omega_{LW}} = \frac{k_L}{4}\Bigg\{ & A_{LW}^2\left(2\omega_{LW}-\delta_{LW}\right) + \left(\frac{2\omega_{LW}^3 - 7\delta_{LW}\omega_{LW}^2 + 4\omega_{LW}\delta_{LW}^2}{2\left(\omega_{LW}-2\delta_{LW}\right)^2}\right) \\ & * \left(A_{LW}^2 + \left(A_{LW}+\Delta A_{LW}\right)^2\right)\Bigg\}\end{aligned} \tag{8}$$

$$\frac{\partial \overline{F}_{xLW}^{B}}{\partial \eta_{LW}} = \frac{k_L\omega_{LW}\left(\omega_{LW}+\sigma_{LW}\right)}{4}\left(A_{LW}+\Delta A_{LW}\right)\frac{\partial \Delta A_{LW}}{\partial \eta_{LW}} \tag{9}$$

$$\frac{\partial \overline{F}_{xLW}^{B}}{\partial \eta_{LW-1}} = \frac{k_L\omega_{LW}\left(\omega_{LW}+\sigma_{LW}\right)}{4}\left(A_{LW}+\Delta A_{LW}\right)\frac{\partial \Delta A_{LW}}{\partial \eta_{LW-1}} \tag{10}$$

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B. Y-Body Axis Force Control Derivatives

$$\begin{aligned}
\frac{\partial \overline{F}_{y\,RW}^{B}}{\partial \delta_{RW}} = & \frac{k_D A_{RW} J_1(A_{RW}) \omega_{RW} \sin \eta_{RW}}{2} + \frac{k_D}{4} \left(\frac{\omega_{RW}^3}{\left(\omega_{RW} - 2\delta_{RW}\right)^2} \right) \\
& * \Bigg\{ A_{RW} \left[J_1(A_{RW}) \sin \eta_{RW} - H_1(A_{RW}) \cos \eta_{RW} \right] \\
& + \left(A_{RW} + \Delta A_{RW}\right) \Bigg[J_1(A_{RW} + \Delta A_{RW}) \sin \eta_{RW} \\
& + \; H_1(A_{RW} + \Delta A_{RW}) \cos \eta_{RW} \Bigg] \Bigg\}
\end{aligned} \tag{11}$$

$$\begin{aligned}
\frac{\partial \overline{F}_{y\,RW}^{B}}{\partial \omega_{RW}} = & \frac{-k_D A_{RW} J_1(A_{RW}) \left(2\omega_{RW} - \delta_{RW}\right) \sin \eta_{RW}}{2} \\
& + \frac{k_D}{4} \left(\frac{2\omega_{RW}^3 - 7\delta_{RW}\omega_{RW}^2 + 4\omega_{RW}\delta_{RW}^2}{\left(\omega_{RW} - 2\delta_{RW}\right)^2} \right) \\
& * \Bigg\{ A_{RW} \left[J_1(A_{RW}) \sin \eta_{RW} - H_1(A_{RW}) \cos \eta_{RW} \right] \\
& + \left(A_{RW} + \Delta A_{RW}\right) \Bigg[J_1(A_{RW} + \Delta A_{RW}) \sin \eta_{RW} \\
& + \; H_1(A_{RW} + \Delta A_{RW}) \cos \eta_{RW} \Bigg] \Bigg\}
\end{aligned} \tag{12}$$

$$\begin{aligned}
\frac{\partial \overline{F}_{y\,RW}^{B}}{\partial \eta_{RW}} = & \frac{-k_D A_{RW} J_1(A_{RW}) \omega_{RW} \left(\omega_{RW} - \delta_{RW}\right) \cos \eta_{RW}}{2} \\
& + \frac{k_D A_{RW} \omega_{RW} \left(\omega_{RW} + \sigma_{RW}\right)}{4} \Bigg[J_1(A_{RW}) \cos \eta_{RW} + H_1(A_{RW}) \sin \eta_{RW} \Bigg] \\
& + \frac{k_D \omega_{RW} \left(\omega_{RW} + \sigma_{RW}\right)}{4} \Bigg[\Bigg\{ J_1(A_{RW} + \Delta A_{RW}) \sin \eta_{RW} \\
& + \; H_1(A_{RW} + \Delta A_{RW}) \cos \eta_{RW} \Bigg\} \frac{\partial \Delta A_{RW}}{\partial \eta_{RW}} \\
& + \left(A_{RW} + \Delta A_{RW}\right) \Bigg\{ \frac{\partial J_1(A_{RW} + \Delta A_{RW})}{\partial \eta_{RW}} \sin \eta_{RW} + J_1(A_{RW} + \Delta A_{RW}) \cos \eta_{RW} \\
& + \; \frac{\partial H_1(A_{RW} + \Delta A_{RW})}{\partial \eta_{RW}} \cos \eta_{RW} - H_1(A_{RW} + \Delta A_{RW}) \sin \eta_{RW} \Bigg\} \Bigg]
\end{aligned} \tag{13}$$

$$\begin{aligned}
\frac{\partial \overline{F}_{y\,RW}^{B}}{\partial \eta_{RW-1}} = & \frac{k_D \omega_{RW} \left(\omega_{RW} + \sigma_{RW}\right)}{4} \Bigg[\Bigg\{ J_1(A_{RW} + \Delta A_{RW}) \sin \eta_{RW} \\
& + \; H_1(A_{RW} + \Delta A_{RW}) \cos \eta_{RW} \Bigg\} \frac{\partial \Delta A_{RW}}{\partial \eta_{RW-1}} \\
& + \left(A_{RW} + \Delta A_{RW}\right) \Bigg\{ \frac{\partial J_1(A_{RW} + \Delta A_{RW})}{\partial \eta_{RW-1}} \sin \eta_{RW} \\
& + \; \frac{\partial H_1(A_{RW} + \Delta A_{RW})}{\partial \eta_{RW-1}} \cos \eta_{RW} \Bigg\} \Bigg]
\end{aligned} \tag{14}$$

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where $J_1(\cdot)$ is a Bessel function of the first kind and $H_1(\cdot)$ is a Struve function.[5]

$$
\begin{aligned}
\frac{\partial \overline{F}^B_{yLW}}{\partial \delta_{LW}} = & \frac{-k_D A_{LW} J_1(A_{LW}) \omega_{LW} \sin \eta_{LW}}{2} - \frac{k_D}{4}\left(\frac{\omega^3_{LW}}{(\omega_{LW} - 2\delta_{LW})^2}\right) \\
& * \Big\{ A_{LW} \left[J_1(A_{LW}) \sin \eta_{LW} - H_1(A_{LW}) \cos \eta_{LW} \right] \\
& + (A_{LW} + \Delta A_{LW}) \left[J_1(A_{LW} + \Delta A_{LW}) \sin \eta_{LW} \right. \\
& \left. + \; H_1 \left(A_{LW} + \Delta A_{LW} \right) \cos \eta_{LW} \right] \Big\}
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
\frac{\partial \overline{F}^B_{yLW}}{\partial \omega_{LW}} = & \frac{k_D A_{LW} J_1(A_{LW}) \left(2\omega_{LW} - \delta_{LW}\right) \sin \eta_{LW}}{2} \\
& - \frac{k_D}{4}\left(\frac{2\omega^3_{LW} - 7\delta_{LW}\omega^2_{LW} + 4\omega_{LW}\delta^2_{LW}}{(\omega_{LW} - 2\delta_{LW})^2}\right) \\
& * \Big\{ A_{LW} \left[J_1(A_{LW}) \sin \eta_{LW} - H_1(A_{LW}) \cos \eta_{LW} \right] \\
& + (A_{LW} + \Delta A_{LW}) \left[J_1(A_{LW} + \Delta A_{LW}) \sin \eta_{LW} \right. \\
& \left. + \; H_1 \left(A_{LW} + \Delta A_{LW} \right) \cos \eta_{LW} \right] \Big\}
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
\frac{\partial \overline{F}^B_{yLW}}{\partial \eta_{LW}} = & \frac{k_D A_{LW} J_1(A_{LW}) \omega_{LW} \left(\omega_{LW} - \delta_{LW}\right) \cos \eta_{LW}}{2} \\
& - \frac{k_D A_{LW} \omega_{LW} \left(\omega_{LW} + \sigma_{LW}\right)}{4} \Bigg[J_1(A_{LW}) \cos \eta + H_1(A_{LW}) \sin \eta_{LW} \Bigg] \\
& - \frac{k_D \omega_{LW} \left(\omega_{LW} + \sigma_{LW}\right)}{4} \Bigg[\Big\{ J_1(A_{LW} + \Delta A_{LW}) \sin \eta_{LW} \\
& + \; H_1(A_{LW} + \Delta A_{LW}) \cos \eta \Big\} \frac{\partial \Delta A_{LW}}{\partial \eta_{LW}} \\
& + (A_{LW} + \Delta A_{LW}) \Big\{ \frac{\partial J_1(A_{LW} + \Delta A_{LW})}{\partial \eta_{LW}} \sin \eta_{LW} + J_1(A_{LW} + \Delta A_{LW}) \cos \eta_{LW} \\
& + \; \frac{\partial H_1(A_{LW} + \Delta A_{LW})}{\partial \eta_{LW}} \cos \eta_{LW} - H_1(A_{LW} + \Delta A_{LW}) \sin \eta_{LW} \Big\} \Bigg]
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
\frac{\partial \overline{F}^B_{yLW}}{\partial \eta_{LW-1}} = & - \frac{k_D \omega_{LW} \left(\omega_{LW} + \sigma_{LW}\right)}{4} \Bigg[\Big\{ J_1(A_{LW} + \Delta A_{LW}) \sin \eta_{LW} \\
& + \; H_1(A_{LW} + \Delta A_{LW}) \cos \eta_{LW} \Big\} \frac{\partial \Delta A_{LW}}{\partial \eta_{LW-1}} \\
& + (A_{LW} + \Delta A_{LW}) \Big\{ \frac{\partial J_1(A_{LW} + \Delta A_{LW})}{\partial \eta_{LW-1}} \sin \eta_{LW} \\
& + \; \frac{\partial H_1(A_{LW} + \Delta A_{LW})}{\partial \eta_{LW-1}} \cos \eta_{LW} \Big\} \Bigg]
\end{aligned} \tag{18}
$$

C. Z-Body Axis Force Control Derivatives

$$\frac{\partial \overline{F}_{z\,RW}^{B}}{\partial \delta_{RW}} = -\frac{k_D A_{RW} J_1(A_{RW}) \omega_{RW} \cos\eta_{RW}}{2} - \frac{k_D}{4}\left(\frac{\omega_{RW}^3}{(\omega_{RW}-2\delta_{RW})^2}\right)$$
$$* \left\{ A_{RW}\left[J_1(A_{RW})\cos\eta_{RW} + H_1(A_{RW})\sin\eta_{RW}\right]\right.$$
$$+ (A_{RW}+\Delta A_{RW})\left[J_1(A_{RW}+\Delta A_{RW})\cos\eta_{RW}\right.$$
$$\left.\left. - H_1(A_{RW}+\Delta A_{RW})\sin\eta_{RW}\right]\right\} \tag{19}$$

$$\frac{\partial \overline{F}_{z\,RW}^{B}}{\partial \omega_{RW}} = \frac{k_D A_{RW} J_1(A_{RW})(2\omega_{RW}-\delta_{RW})\cos\eta_{RW}}{2} - \frac{k_D}{4}\left(\frac{2\omega_{RW}^3 - 7\delta_{RW}\omega_{RW}^2 + 4\omega_{RW}\delta_{RW}^2}{(\omega_{RW}-2\delta_{RW})^2}\right)$$
$$* \left\{ A_{RW}\left[J_1(A_{RW})\cos\eta_{RW} + H_1(A_{RW})\sin\eta_{RW}\right]\right.$$
$$+ (A_{RW}+\Delta A_{RW})\left[J_1(A_{RW}+\Delta A_{RW})\cos\eta_{RW}\right.$$
$$\left.\left. - H_1(A_{RW}+\Delta A_{RW})\sin\eta_{RW}\right]\right\} \tag{20}$$

$$\frac{\partial \overline{F}_{z\,RW}^{B}}{\partial \eta_{RW}} = \frac{-k_D A_{RW} J_1(A_{RW})\omega_{RW}(\omega_{RW}-\delta_{RW})\sin\eta_{RW}}{2}$$
$$-\frac{k_D A_{RW}\omega_{RW}(\omega_{RW}+\sigma_{RW})}{4}\left[-J_1(A_{RW})\sin\eta_{RW} + H_1(A_{RW})\cos\eta_{RW}\right]$$
$$-\frac{k_D\omega_{RW}(\omega_{RW}+\sigma_{RW})}{4}\left[\left\{J_1(A_{RW}+\Delta A_{RW})\cos\eta_{RW}\right.\right.$$
$$\left. - H_1(A_{RW}+\Delta A_{RW})\sin\eta_{RW}\right\}\frac{\partial \Delta A_{RW}}{\partial \eta_{RW}}$$
$$+ (A_{RW}+\Delta A_{RW})\left\{\frac{\partial J_1(A_{RW}+\Delta A_{RW})}{\partial \eta_{RW}}\cos\eta_{RW} - J_1(A_{RW}+\Delta A_{RW})\sin\eta_{RW}\right.$$
$$\left.\left. - \frac{\partial H_1(A_{RW}+\Delta A_{RW})}{\partial \eta_{RW}}\sin\eta_{RW} - H_1(A_{RW}+\Delta A_{RW})\cos\eta_{RW}\right\}\right] \tag{21}$$

$$\frac{\partial \overline{F}_{z\,RW}^{B}}{\partial \eta_{RW-1}} = -\frac{k_D\omega_{RW}(\omega_{RW}+\sigma_{RW})}{4}\left[\left\{J_1(A_{RW}+\Delta A_{RW})\cos\eta_{RW}\right.\right.$$
$$\left. - H_1(A_{RW}+\Delta A_{RW})\sin\eta_{RW}\right\}\frac{\partial \Delta A_{RW}}{\partial \eta_{RW-1}}$$
$$+ (A_{RW}+\Delta A_{RW})\left\{\frac{\partial J_1(A_{RW}+\Delta A_{RW})}{\partial \eta_{RW-1}}\cos\eta_{RW}\right.$$
$$\left.\left. - \frac{\partial H_1(A_{RW}+\Delta A_{RW})}{\partial \eta_{RW-1}}\sin\eta_{RW}\right\}\right] \tag{22}$$

$$
\begin{aligned}
\frac{\partial \overline{F_z}^B_{LW}}{\partial \delta_{LW}} = &-\frac{k_D A_{LW} J_1(A_{LW}) \omega_{LW} \cos\eta_{LW}}{2} - \frac{k_D}{4}\left(\frac{\omega_{LW}^3}{\left(\omega_{LW}-2\delta_{LW}\right)^2}\right) \\
&* \Big\{ A_{LW}\left[J_1(A_{LW})\cos\eta_{LW} + H_1(A_{LW})\sin\eta_{LW}\right] \\
&+ \left(A_{LW}+\Delta A_{LW}\right) \Big[J_1(A_{LW}+\Delta A_{LW})\cos\eta_{LW} \\
&- H_1\left(A_{LW}+\Delta A_{LW}\right)\sin\eta_{LW} \Big] \Big\}
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
\frac{\partial \overline{F_z}^B_{LW}}{\partial \omega_{LW}} = &\frac{k_D A_{LW} J_1(A_{LW}) \left(2\omega_{LW}-\delta_{LW}\right) \cos\eta_{LW}}{2} - \frac{k_D}{4}\left(\frac{2\omega_{LW}^3 - 7\delta_{LW}\omega_{LW}^2 + 4\omega_{LW}\delta_{LW}^2}{\left(\omega_{LW}-2\delta_{LW}\right)^2}\right) \\
&* \Big\{ A_{LW}\left[J_1(A_{LW})\cos\eta_{LW} + H_1(A_{LW})\sin\eta_{LW}\right] \\
&+ \left(A_{LW}+\Delta A_{LW}\right) \Big[J_1(A_{LW}+\Delta A_{LW})\cos\eta_{LW} \\
&- H_1\left(A_{LW}+\Delta A_{LW}\right)\sin\eta_{LW} \Big] \Big\}
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
\frac{\partial \overline{F_z}^B_{LW}}{\partial \eta_{LW}} = &\frac{-k_D A_{LW} J_1(A_{LW}) \omega_{LW}\left(\omega_{LW}-\delta_{LW}\right) \sin\eta_{LW}}{2} \\
&-\frac{k_D A_{LW}\omega_{LW}\left(\omega_{LW}+\sigma_{LW}\right)}{4}\left[-J_1(A_{LW})\sin\eta_{LW} + H_1(A_{LW})\cos\eta_{LW}\right] \\
&-\frac{k_D\omega_{LW}\left(\omega_{LW}+\sigma_{LW}\right)}{4}\Bigg[\Big\{ J_1(A_{LW}+\Delta A_{LW})\cos\eta_{LW} \\
&- H_1(A_{LW}+\Delta A_{LW})\sin\eta_{LW} \Big\} \frac{\partial \Delta A_{LW}}{\partial \eta_{LW}} \\
&+ \left(A_{LW}+\Delta A_{LW}\right)\Big\{ \frac{\partial J_1(A_{LW}+\Delta A_{LW})}{\partial \eta_{LW}}\cos\eta_{LW} - J_1(A_{LW}+\Delta A_{LW})\sin\eta_{LW} \\
&- \frac{\partial H_1(A_{LW}+\Delta A_{LW})}{\partial \eta_{LW}}\sin\eta_{LW} - H_1(A_{LW}+\Delta A_{LW})\cos\eta_{LW} \Big\} \Bigg]
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
\frac{\partial \overline{F_z}^B_{LW}}{\partial \eta_{LW-1}} = &-\frac{k_D\omega_{LW}\left(\omega_{LW}+\sigma_{LW}\right)}{4}\Bigg[\Big\{ J_1(A_{LW}+\Delta A_{LW})\cos\eta_{LW} \\
&- H_1(A_{LW}+\Delta A_{LW})\sin\eta_{LW} \Big\} \frac{\partial \Delta A_{LW}}{\partial \eta_{LW-1}} \\
&+ \left(A_{LW}+\Delta A_{LW}\right)\Big\{ \frac{\partial J_1(A_{LW}+\Delta A_{LW})}{\partial \eta_{LW-1}}\cos\eta_{LW} \\
&- \frac{\partial H_1(A_{LW}+\Delta A_{LW})}{\partial \eta_{LW-1}}\sin\eta_{LW} \Big\} \Bigg]
\end{aligned}
\tag{26}
$$

D. Rolling Moment Control Derivatives

The rolling moment control derivatives are

$$\begin{aligned}\frac{\partial \overline{M}_{x\,RW}^{B}}{\partial \delta_{RW}} &= -\frac{k_D A_{RW}\omega_{RW}}{4} RM_{1_{RW}} \\ &\quad - \frac{k_D}{8}\left[A_{RW} RM_{2_{RW}} + \left(A_{RW} + \Delta A_{RW}\right) RM_{3_{RW}}\right] \frac{\omega_{RW}^3}{\left(\omega_{RW} - 2\delta_{RW}\right)^2}\end{aligned} \tag{27}$$

where

$$RM_{1_{RW}} = y_{cp}^{WP} A_{RW} + J_1(A_{RW}) \left\{w \cos\eta_{RW} + 2\Delta z_R^B \sin\eta_{RW}\right\} \tag{28}$$

$$\begin{aligned}RM_{2_{RW}} &= y_{cp}^{WP} A_{RW} + w\left\{J_1(A_{RW})\cos\eta_{RW} + H_1(A_{RW})\sin\eta_{RW}\right\} \\ &\quad +2\Delta z_R^B \left\{J_1(A_{RW})\sin\eta_{RW} - H_1(A_{RW})\cos\eta_{RW}\right\}\end{aligned} \tag{29}$$

$$\begin{aligned}RM_{3_{RW}} &= y_{cp}^{WP}\left(A_{RW} + \Delta A_{RW}\right) + w\left\{J_1\left(A_{RW} + \Delta A_{RW}\right)\cos\eta_{RW}\right. \\ &\quad \left. -H_1\left(A_{RW} + \Delta A_{RW}\right)\sin\eta_{RW}\right\} \\ &\quad +2\Delta z_R^B\left\{J_1\left(A_{RW} + \Delta A_{RW}\right)\sin\eta_{RW} + H_1\left(A_{RW} + \Delta A_{RW}\right)\cos\eta_{RW}\right\}\end{aligned} \tag{30}$$

$$\frac{\partial \overline{M}_{x\,RW}^{B}}{\partial \omega_{RW}} = \frac{k_D A_{RW}\left(2\omega_{RW} - \delta_{RW}\right)}{4} RM_{1_{RW}} - \frac{k_D}{8}\left[A_{RW} RM_{2_{RW}} + \left(A_{RW} + \Delta A_{RW}\right) RM_{3_{RW}}\right] * \frac{\left(2\omega_{RW}^3 - 7\delta_{RW}\omega_{RW}^2 + 4\omega_{RW}\delta_{RW}^2\right)}{\left(\omega_{RW} - 2\delta_{RW}\right)^2} \tag{31}$$

$$\begin{aligned}
\frac{\partial \overline{M}_{x\,RW}^{B}}{\partial \eta_{RW}} = {} & \frac{k_D A_{RW}\omega_{RW}\left(\omega_{RW} - \delta_{RW}\right)}{4}\left[J_1(A_{RW})\left\{-w \sin\eta_{RW} + 2\Delta z_R^B \cos\eta_{RW}\right\}\right] \\
& - \frac{k_D A_{RW}\omega_{RW}\left(\omega_{RW} + \sigma_{RW}\right)}{8}\Bigg[w\left\{-J_1(A_{RW})\sin\eta_{RW} + H_1(A_{RW})\cos\eta_{RW}\right\} \\
& + 2\Delta z_R^B\left\{J_1(A_{RW})\cos\eta_{RW} + H_1(A_{RW})\sin\eta_{RW}\right\}\Bigg] \\
& - \frac{k_D \omega_{RW}\left(\omega_{RW} + \sigma_{RW}\right)}{8}\frac{\partial \Delta A_{RW}}{\partial \eta_{RW}}\Bigg[y_{cp}^{WP}\left(A_{RW} + \Delta A_{RW}\right) \\
& + w\left\{J_1\left(A_{RW} + \Delta A_{RW}\right)\cos\eta_{RW} - H_1\left(A_{RW} + \Delta A_{RW}\right)\sin\eta_{RW}\right\}\Bigg] \\
& - \frac{k_D \omega_{RW}\left(\omega_{RW} + \sigma_{RW}\right)}{8}\left(A_{RW} + \Delta A_{RW}\right)\Bigg[y_{cp}^{WP}\frac{\partial \Delta A_{RW}}{\partial \eta_{RW}} \\
& + w\Bigg\{\frac{\partial J_1(A_{RW} + \Delta A_{RW})}{\partial \eta_{RW}}\cos\eta_{RW} - J_1\left(A_{RW} + \Delta A_{RW}\right)\sin\eta_{RW} \\
& - \frac{\partial H_1(A_{RW} + \Delta A_{RW})}{\partial \eta_{RW}}\sin\eta_{RW} - H_1\left(A_{RW} + \Delta A_{RW}\right)\cos\eta_{RW}\Bigg\}\Bigg] \\
& - \frac{k_D \omega_{RW}\left(\omega_{RW} + \sigma_{RW}\right)}{8}\frac{\partial \Delta A_{RW}}{\partial \eta_{RW}}\Bigg[2\Delta z_R^B\Bigg\{J_1\left(A_{RW} + \Delta A_{RW}\right)\sin\eta_{RW} \\
& + H_1\left(A_{RW} + \Delta A_{RW}\right)\cos\eta_{RW}\Bigg\}\Bigg] \\
& - \frac{k_D \omega_{RW}\left(\omega_{RW} + \sigma_{RW}\right)}{8}\left(A_{RW} + \Delta A_{RW}\right)\Bigg[2\Delta z_R^B\Bigg\{\frac{\partial J_1(A_{RW} + \Delta A_{RW})}{\partial \eta_{RW}}\sin\eta_{RW} \\
& + J_1\left(A_{RW} + \Delta A_{RW}\right)\cos\eta_{RW} + \frac{\partial H_1(A_{RW} + \Delta A_{RW})}{\partial \eta_{RW}}\cos\eta_{RW} \\
& - H_1\left(A_{RW} + \Delta A_{RW}\right)\sin\eta_{RW}\Bigg\}\Bigg]
\end{aligned} \tag{32}$$

$$
\begin{aligned}
\frac{\partial \overline{M}_{x\,RW}^{B}}{\partial \eta_{RW-1}} = &-\frac{k_D \omega_{RW}\left(\omega_{RW}+\sigma_{RW}\right)}{8}\frac{\partial \Delta A_{RW}}{\partial \eta_{RW-1}}\left[y_{cp}^{WP}\left(A_{RW}+\Delta A_{RW}\right)\right.\\
&\left.+w\left\{J_1\left(A_{RW}+\Delta A_{RW}\right)\cos\eta_{RW}-H_1\left(A_{RW}+\Delta A_{RW}\right)\sin\eta_{RW}\right\}\right]\\
&-\frac{k_D \omega_{RW}\left(\omega_{RW}+\sigma_{RW}\right)}{8}\left(A_{RW}+\Delta A_{RW}\right)\left[y_{cp}^{WP}\frac{\partial \Delta A_{RW}}{\partial \eta_{RW-1}}\right.\\
&\left.+\,w\left\{\frac{\partial J_1(A_{RW}+\Delta A_{RW})}{\partial \eta_{RW-1}}\cos\eta_{RW}-\frac{\partial H_1(A_{RW}+\Delta A_{RW})}{\partial \eta_{RW-1}}\sin\eta_{RW}\right\}\right]\\
&-\frac{k_D \omega_{RW}\left(\omega_{RW}+\sigma_{RW}\right)}{8}\frac{\partial \Delta A_{RW}}{\partial \eta_{RW-1}}\left[2\Delta z_R^B\left\{J_1\left(A_{RW}+\Delta A_{RW}\right)\sin\eta_{RW}\right.\right.\\
&\left.\left.+\;H_1\left(A_{RW}+\Delta A_{RW}\right)\cos\eta_{RW}\right\}\right]\\
&-\frac{k_D \omega_{RW}\left(\omega_{RW}+\sigma_{RW}\right)}{8}\left(A_{RW}+\Delta A_{RW}\right)\left[2\Delta z_R^B\left\{\frac{\partial J_1(A_{RW}+\Delta A_{RW})}{\partial \eta_{RW-1}}\sin\eta_{RW}\right.\right.\\
&\left.\left.+\frac{\partial H_1(A_{RW}+\Delta A_{RW})}{\partial \eta_{RW-1}}\cos\eta_{RW}\right\}\right] \qquad (33)
\end{aligned}
$$

For the left wing, the rolling moments control derivatives are

$$
\begin{aligned}
\frac{\partial \overline{M}_{x\,LW}^{B}}{\partial \delta_{LW}} = &\frac{k_D A_{LW}\omega_{LW}}{4}RM_{1_{LW}}\\
&+\frac{k_D}{8}\left[A_{LW}RM_{2_{LW}}+\left(A_{LW}+\Delta A_{LW}\right)RM_{3_{LW}}\right]\frac{\omega_{LW}^3}{\left(\omega_{LW}-2\delta_{LW}\right)^2}
\end{aligned} \qquad (34)
$$

where

$$
RM_{1_{LW}} = y_{cp}^{WP}A_{LW}+J_1(A_{LW})\left\{w\cos\eta_{LW}+2\Delta z_L^B\sin\eta_{LW}\right\} \qquad (35)
$$

$$
\begin{aligned}
RM_{2_{LW}} = &\,y_{cp}^{WP}A_{LW}+w\left\{J_1(A_{LW})\cos\eta_{LW}+H_1(A_{LW})\sin\eta_{LW}\right\}\\
&+2\Delta z_L^B\left\{J_1(A_{LW})\sin\eta_{LW}-H_1(A_{LW})\cos\eta_{LW}\right\}
\end{aligned} \qquad (36)
$$

$$
\begin{aligned}
RM_{3_{LW}} = &\,y_{cp}^{WP}\left(A_{LW}+\Delta A_{LW}\right)+w\left\{J_1\left(A_{LW}+\Delta A_{LW}\right)\cos\eta_{LW}\right.\\
&\left.-H_1\left(A_{LW}+\Delta A_{LW}\right)\sin\eta_{LW}\right\}\\
&+2\Delta z_L^B\left\{J_1\left(A_{LW}+\Delta A_{LW}\right)\sin\eta_{LW}+H_1\left(A_{LW}+\Delta A_{LW}\right)\cos\eta_{LW}\right\}
\end{aligned} \qquad (37)
$$

$$\frac{\partial \overline{M_x}^B_{LW}}{\partial \omega_{LW}} = -\frac{k_D A_{LW}\left(2\omega_{LW}-\delta_{LW}\right)}{4} RM_{1_{LW}} + \frac{k_D}{8}\left[A_{LW} RM_{2_{LW}} + \left(A_{LW}+\Delta A_{LW}\right) RM_{3_{LW}}\right]$$
$$* \frac{\left(2\omega_{LW}^3 - 7\delta_{LW}\omega_{LW}^2 + 4\omega_{LW}\delta_{LW}^2\right)}{\left(\omega_{LW}-2\delta_{LW}\right)^2} \tag{38}$$

$$\frac{\partial \overline{M_x}^B_{LW}}{\partial \eta_{LW}} = -\frac{k_D A_{LW}\omega_{LW}\left(\omega_{LW}-\delta_{LW}\right)}{4}\left[J_1(A_{LW})\left\{-w\sin\eta_{LW} + 2\Delta z_L^B \cos\eta_{LW}\right\}\right]$$
$$+\frac{k_D A_{LW}\omega_{LW}\left(\omega_{LW}+\sigma_{LW}\right)}{8}\Bigg[w\left\{-J_1(A_{LW})\sin\eta_{LW} + H_1(A_{LW})\cos\eta_{LW}\right\}$$
$$+ 2\Delta z_L^B\left\{J_1(A_{LW})\cos\eta_{LW} + H_1(A_{LW})\sin\eta_{LW}\right\}\Bigg]$$
$$+\frac{k_D\omega_{LW}\left(\omega_{LW}+\sigma_{LW}\right)}{8}\frac{\partial \Delta A_{LW}}{\partial \eta_{LW}}\Bigg[y_{cp}^{WP}\left(A_{LW}+\Delta A_{LW}\right)$$
$$+ w\left\{J_1\left(A_{LW}+\Delta A_{LW}\right)\cos\eta_{LW} - H_1\left(A_{LW}+\Delta A_{LW}\right)\sin\eta_{LW}\right\}\Bigg]$$
$$+\frac{k_D\omega_{LW}\left(\omega_{LW}+\sigma_{LW}\right)}{8}\left(A_{LW}+\Delta A_{LW}\right)\Bigg[y_{cp}^{WP}\frac{\partial \Delta A_{LW}}{\partial \eta_{LW}}$$
$$+ w\Bigg\{\frac{\partial J_1(A_{LW}+\Delta A_{LW})}{\partial \eta_{LW}}\cos\eta_{LW} - J_1\left(A_{LW}+\Delta A_{LW}\right)\sin\eta_{LW}$$
$$-\frac{\partial H_1(A_{LW}+\Delta A_{LW})}{\partial \eta_{LW}}\sin\eta_{LW} - H_1\left(A_{LW}+\Delta A_{LW}\right)\cos\eta_{LW}\Bigg\}\Bigg]$$
$$+\frac{k_D\omega_{LW}\left(\omega_{LW}+\sigma_{LW}\right)}{8}\frac{\partial \Delta A_{LW}}{\partial \eta_{LW}}\Bigg[2\Delta z_L^B\Bigg\{J_1\left(A_{LW}+\Delta A_{LW}\right)\sin\eta_{LW}$$
$$+ H_1\left(A_{LW}+\Delta A_{LW}\right)\cos\eta_{LW}\Bigg\}\Bigg]$$
$$+\frac{k_D\omega_{LW}\left(\omega_{LW}+\sigma_{LW}\right)}{8}\left(A_{LW}+\Delta A_{LW}\right)\Bigg[2\Delta z_L^B\Bigg\{\frac{\partial J_1(A_{LW}+\Delta A_{LW})}{\partial \eta_{LW}}\sin\eta_{LW}$$
$$+J_1\left(A_{LW}+\Delta A_{LW}\right)\cos\eta_{LW} + \frac{\partial H_1(A_{LW}+\Delta A_{LW})}{\partial \eta_{LW}}\cos\eta_{LW}$$
$$- H_1\left(A_{LW}+\Delta A_{LW}\right)\sin\eta_{LW}\Bigg\}\Bigg] \tag{39}$$

$$
\begin{aligned}
\frac{\partial \overline{M}_{xLW}^{B}}{\partial \eta_{LW-1}} = & \frac{k_D \omega_{LW}\left(\omega_{LW}+\sigma_{LW}\right)}{8} \frac{\partial \Delta A_{LW}}{\partial \eta_{LW-1}}\left[y_{cp}^{WP}\left(A_{LW}+\Delta A_{LW}\right)\right. \\
& \left. + w\left\{J_1\left(A_{LW}+\Delta A_{LW}\right)\cos\eta_{LW} - H_1\left(A_{LW}+\Delta A_{LW}\right)\sin\eta_{LW}\right\}\right] \\
& + \frac{k_D \omega_{LW}\left(\omega_{LW}+\sigma_{LW}\right)}{8}\left(A_{LW}+\Delta A_{LW}\right)\left[y_{cp}^{WP}\frac{\partial \Delta A_{LW}}{\partial \eta_{LW-1}}\right. \\
& \left. + w\left\{\frac{\partial J_1(A_{LW}+\Delta A_{LW})}{\partial \eta_{LW-1}}\cos\eta_{LW} - \frac{\partial H_1(A_{LW}+\Delta A_{LW})}{\partial \eta_{LW-1}}\sin\eta_{LW}\right\}\right] \\
& + \frac{k_D \omega_{LW}\left(\omega_{LW}+\sigma_{LW}\right)}{8}\frac{\partial \Delta A_{LW}}{\partial \eta_{LW-1}}\left[2\Delta z_L^B\left\{J_1\left(A_{LW}+\Delta A_{LW}\right)\sin\eta_{LW}\right.\right. \\
& \left.\left. + \ H_1\left(A_{LW}+\Delta A_{LW}\right)\cos\eta_{LW}\right\}\right] \\
& + \frac{k_D \omega_{LW}\left(\omega_{LW}+\sigma_{LW}\right)}{8}\left(A_{LW}+\Delta A_{LW}\right)\left[2\Delta z_L^B\left\{\frac{\partial J_1(A_{LW}+\Delta A_{LW})}{\partial \eta_{LW-1}}\sin\eta_{LW}\right.\right. \\
& \left.\left. + \frac{\partial H_1(A_{LW}+\Delta A_{LW})}{\partial \eta_{LW-1}}\cos\eta_{LW}\right\}\right] \qquad (40)
\end{aligned}
$$

E. Pitching Moment Control Derivatives

The pitching moment control derivatives are

$$
\begin{aligned}
\frac{\partial \overline{M}_{yRW}^{B}}{\partial \delta_{RW}} = & \frac{A_{RW}\omega_{RW}}{2} PM_{1_{RW}} + \left(\frac{PM_{2_{RW}}}{4A_{RW}} + \frac{y_{cp}^{WP}k_L}{4A_{RW}}PM_{3_{RW}}\right. \\
& \left. + \frac{\Delta z_R^B k_L}{8}\left(A_{RW}^2 + \left(A_{RW}+\Delta A_{RW}\right)^2\right)\right)\left(\frac{\omega_{RW}^3}{\left(\omega_{RW}-2\delta_{RW}\right)^2}\right) \qquad (41)
\end{aligned}
$$

where

$$
\begin{aligned}
PM_{1_{RW}} = & \left(\cos\alpha x_{cp}^{WP} k_L + \left\{\sin\alpha x_{cp}^{WP} + \Delta x_R^B\right\} k_D\right) J_1(A_{RW})\cos\eta_{RW} \\
& - y_{cp}^{WP} k_L J_1(A_{RW})\sin\eta_{RW} - \frac{\Delta z_R^B}{2} k_L A_{RW} \qquad (42) \\
PM_{2_{RW}} = & \left(\cos\alpha x_{cp}^{WP} k_L + \left\{\sin\alpha x_{cp}^{WP} + \Delta x_R^B\right\} k_D\right)\left[A_{RW}^2\left\{J_1(A_{RW})\cos\eta_{RW}\right.\right. \\
& \left. + \ H_1(A_{RW})\sin\eta_{RW}\right\} + A_{RW}\left(A_{RW}+\Delta A_{RW}\right)\left\{J_1(A_{RW}+\Delta A_{RW})\cos\eta_{RW}\right. \\
& \left.\left. - \ H_1(A_{RW}+\Delta A_{RW})\sin\eta_{RW}\right\}\right] \qquad (43) \\
PM_{3_{RW}} = & A_{RW}^2\left\{J_1(A_{RW})\sin\eta_{RW} - H_1(A_{RW})\cos\eta_{RW}\right\} + A_{RW}\left(A_{RW}+\Delta A_{RW}\right) \\
& * \left\{J_1(A_{RW}+\Delta A_{RW})\sin\eta_{RW} + H_1(A_{RW}+\Delta A_{RW})\cos\eta_{RW}\right\} \qquad (44)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \overline{M}_{y\,RW}^{B}}{\partial \omega_{RW}} = {} & \frac{-A_{RW}\left(2\omega_{RW}-\delta_{RW}\right)}{2} PM_{1_{RW}} + \left(\frac{PM_{2_{RW}}}{4A_{RW}} + \frac{y_{cp}^{WP} k_L}{4A_{RW}} PM_{3_{RW}} \right. \\
& \left. + \frac{\Delta z_R^B k_L}{8}\left(A_{RW}^2 + \left(A_{RW}+\Delta A_{RW}\right)^2\right) \right) \left(\frac{2\omega_{RW}^3 - 7\delta_{RW}\omega_{RW}^2 + 4\omega_{RW}\delta_{RW}^2}{\left(\omega_{RW}-2\delta_{RW}\right)^2} \right)
\end{aligned} \tag{45}
$$

$$
\begin{aligned}
\frac{\partial \overline{M}_{y\,RW}^{B}}{\partial \eta_{RW}} = {} & \frac{-A_{RW}\omega_{RW}\left(\omega_{RW}-\delta_{RW}\right)}{2} \Big[- \left(\cos\alpha x_{cp}^{WP} k_L + \left\{\sin\alpha x_{cp}^{WP} + \Delta x_R^B\right\} k_D\right) \\
& * J_1(A_{RW}) \sin\eta_{RW} - y_{cp}^{WP} k_L J_1(A_{RW}) \cos\eta_{RW} \Big] \\
& + \frac{\omega_{RW}\left(\omega_{RW}+\sigma_{RW}\right)}{4A_{RW}} \left(\cos\alpha x_{cp}^{WP} k_L + \left\{\sin\alpha x_{cp}^{WP} + \Delta x_R^B\right\} k_D\right) \\
& * \Big[A_{RW}^2 \left\{-J_1(A_{RW})\sin\eta_{RW} + H_1(A_{RW})\cos\eta_{RW}\right\} \\
+ \quad & A_{RW} \frac{\partial \Delta A_{RW}}{\partial \eta_{RW}} \left\{J_1\left(A_{RW}+\Delta A_{RW}\right)\cos\eta_{RW} - H_1\left(A_{RW}+\Delta A_{RW}\right)\sin\eta_{RW}\right\} \\
& + A_{RW}\left(A_{RW}+\Delta A_{RW}\right) \Big\{ - J_1(A_{RW}+\Delta A_{RW})\sin\eta_{RW} \\
& - \ H_1(A_{RW}+\Delta A_{RW})\cos\eta_{RW} \\
& + \ \frac{\partial J_1\left(A_{RW}+\Delta A_{RW}\right)}{\partial \eta_{RW}} \cos\eta_{RW} - \frac{\partial H_1\left(A_{RW}+\Delta A_{RW}\right)}{\partial \eta_{RW}} \sin\eta_{RW} \Big\} \Big] \\
& + \frac{y_{cp}^{WP} k_L \omega_{RW}\left(\omega_{RW}+\sigma_{RW}\right)}{4A_{RW}} \Big[A_{RW}^2 \left\{J_1(A_{RW})\cos\eta_{RW} + H_1(A_{RW})\sin\eta_{RW}\right\} \\
& + A_{RW} \frac{\partial \Delta A_{RW}}{\partial \eta_{RW}} \left\{J_1\left(A_{RW}+\Delta A_{RW}\right)\sin\eta_{RW} + H_1\left(A_{RW}+\Delta A_{RW}\right)\cos\eta_{RW}\right\} \\
& + A_{RW}\left(A_{RW}+\Delta A_{RW}\right) \Big\{ J_1(A_{RW}+\Delta A_{RW})\cos\eta_{RW} \\
& - \ H_1(A_{RW}+\Delta A_{RW})\sin\eta_{RW} \\
& + \ \frac{\partial J_1\left(A_{RW}+\Delta A_{RW}\right)}{\partial \eta_{RW}} \sin\eta_{RW} + \frac{\partial H_1\left(A_{RW}+\Delta A_{RW}\right)}{\partial \eta_{RW}} \cos\eta_{RW} \Big\} \Big] \\
& + \frac{\Delta z_R^B k_L \omega_{RW}\left(\omega_{RW}+\sigma_{RW}\right)}{4} \left(A_{RW}+\Delta A_{RW}\right) \frac{\partial \Delta A_{RW}}{\partial \eta_{RW}}
\end{aligned} \tag{46}
$$

$$
\begin{aligned}
\frac{\partial \overline{M_y}^B_{RW}}{\partial \eta_{RW-1}} &= \frac{\omega_{RW}\left(\omega_{RW}+\sigma_{RW}\right)}{4A_{RW}}\left(\cos\alpha x_{cp}^{WP}k_L + \left\{\sin\alpha x_{cp}^{WP}+\Delta x_R^B\right\}k_D\right)\\
&* \left[A_{RW}\frac{\partial \Delta A_{RW}}{\partial \eta_{RW-1}}\left\{J_1\left(A_{RW}+\Delta A_{RW}\right)\cos\eta_{RW} - H_1\left(A_{RW}+\Delta A_{RW}\right)\sin\eta_{RW}\right\}\right.\\
&+\ A_{RW}\left(A_{RW}+\Delta A_{RW}\right)\\
&\left.* \left\{\frac{\partial J_1\left(A_{RW}+\Delta A_{RW}\right)}{\partial \eta_{RW-1}}\cos\eta_{RW} - \frac{\partial H_1\left(A_{RW}+\Delta A_{RW}\right)}{\partial \eta_{RW-1}}\sin\eta_{RW}\right\}\right]\\
&+\frac{y_{cp}^{WP}k_L\omega_{RW}\left(\omega_{RW}+\sigma_{RW}\right)}{4A_{RW}}\left[\vphantom{\frac{a}{b}}\right.\\
&+A_{RW}\frac{\partial \Delta A_{RW}}{\partial \eta_{RW-1}}\left\{J_1\left(A_{RW}+\Delta A_{RW}\right)\sin\eta_{RW} + H_1\left(A_{RW}+\Delta A_{RW}\right)\cos\eta_{RW}\right\}\\
&+\ A_{RW}\left(A_{RW}+\Delta A_{RW}\right)\left\{\vphantom{\frac{a}{b}}\right.\\
&\left.\left.+\ \frac{\partial J_1\left(A_{RW}+\Delta A_{RW}\right)}{\partial \eta_{RW-1}}\sin\eta_{RW} + \frac{\partial H_1\left(A_{RW}+\Delta A_{RW}\right)}{\partial \eta_{RW-1}}\cos\eta_{RW}\right\}\right]\\
&+\frac{\Delta z_R^B k_L\omega_{RW}\left(\omega_{RW}+\sigma_{RW}\right)}{4}\left(A_{RW}+\Delta A_{RW}\right)\frac{\partial \Delta A_{RW}}{\partial \eta_{RW-1}} \qquad (47)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \overline{M_y}^B_{LW}}{\partial \delta_{LW}} &= \frac{A_{LW}\omega_{LW}}{2}PM_{1_{LW}} + \left(\frac{PM_{2_{LW}}}{4A_{LW}} + \frac{y_{cp}^{WP}k_L}{4A_{LW}}PM_{3_{LW}}\right.\\
&\left. +\ \frac{\Delta z_L^B k_L}{8}\left(A_{LW}^2 + \left(A_{LW}+\Delta A_{LW}\right)^2\right)\right)\left(\frac{\omega_{LW}^3}{\left(\omega_{LW}-2\delta_{LW}\right)^2}\right) \qquad (48)
\end{aligned}
$$

where

$$
\begin{aligned}
PM_{1_{LW}} &= \left(\cos\alpha x_{cp}^{WP}k_L + \left\{\sin\alpha x_{cp}^{WP}+\Delta x_L^B\right\}k_D\right)J_1(A_{LW})\cos\eta_{LW}\\
&-y_{cp}^{WP}k_L J_1(A_{LW})\sin\eta_{LW} - \frac{\Delta z_L^B}{2}k_L A_{LW} \qquad (49)\\
PM_{2_{LW}} &= \left(\cos\alpha x_{cp}^{WP}k_L + \left\{\sin\alpha x_{cp}^{WP}+\Delta x_L^B\right\}k_D\right)\left[A_{LW}^2\left\{J_1(A_{LW})\cos\eta_{LW}\right.\right.\\
&\left.+\ H_1(A_{LW})\sin\eta_{LW}\right\} + A_{LW}\left(A_{LW}+\Delta A_{LW}\right)\left\{J_1(A_{LW}+\Delta A_{LW})\cos\eta_{LW}\right.\\
&\left.\left.-\ H_1(A_{LW}+\Delta A_{LW})\sin\eta_{LW}\right\}\right] \qquad (50)\\
PM_{3_{LW}} &= A_{LW}^2\left\{J_1(A_{LW})\sin\eta_{LW} - H_1(A_{LW})\cos\eta_{LW}\right\} + A_{LW}\left(A_{LW}+\Delta A_{LW}\right)\\
&*\left\{J_1(A_{LW}+\Delta A_{LW})\sin\eta_{LW} + H_1(A_{LW}+\Delta A_{LW})\cos\eta_{LW}\right\} \qquad (51)
\end{aligned}
$$

$$\frac{\partial \overline{M}_{y\,LW}^{B}}{\partial \omega_{LW}} = \frac{-A_{LW}\left(2\omega_{LW}-\delta_{LW}\right)}{2} PM_{1_{LW}} + \left(\frac{PM_{2_{LW}}}{4A_{LW}} + \frac{y_{cp}^{WP} k_L}{4A_{LW}} PM_{3_{LW}} + \frac{\Delta z_R^B k_L}{8}\left(A_{LW}^2 + \left(A_{LW}+\Delta A_{LW}\right)^2\right)\right)\left(\frac{2\omega_{LW}^3 - 7\delta_{LW}\omega_{LW}^2 + 4\omega_{LW}\delta_{LW}^2}{\left(\omega_{LW}-2\delta_{LW}\right)^2}\right) \tag{52}$$

$$\begin{aligned}
\frac{\partial \overline{M}_{y\,LW}^{B}}{\partial \eta_{LW}} = {} & \frac{-A_{LW}\omega_{LW}\left(\omega_{LW}-\delta_{LW}\right)}{2}\Bigg[-\left(\cos\alpha x_{cp}^{WP} k_L + \left\{\sin\alpha x_{cp}^{WP} + \Delta x_R^B\right\} k_D\right) \\
& * J_1(A_{LW})\sin\eta_{LW} - y_{cp}^{WP} k_L J_1(A_{LW})\cos\eta_{LW}\Bigg] \\
& + \frac{\omega_{LW}\left(\omega_{LW}+\sigma_{LW}\right)}{4A_{LW}}\left(\cos\alpha x_{cp}^{WP} k_L + \left\{\sin\alpha x_{cp}^{WP} + \Delta x_R^B\right\} k_D\right) \\
& * \Bigg[A_{LW}^2\left\{-J_1(A_{LW})\sin\eta_{LW} + H_1(A_{LW})\cos\eta_{LW}\right\} \\
+ \quad & A_{LW}\frac{\partial \Delta A_{LW}}{\partial \eta_{LW}}\left\{J_1\left(A_{LW}+\Delta A_{LW}\right)\cos\eta_{LW} - H_1\left(A_{LW}+\Delta A_{LW}\right)\sin\eta_{LW}\right\} \\
& + A_{LW}\left(A_{LW}+\Delta A_{LW}\right)\Bigg\{-J_1(A_{LW}+\Delta A_{LW})\sin\eta_{LW} \\
& - H_1(A_{LW}+\Delta A_{LW})\cos\eta_{LW} \\
& + \frac{\partial J_1\left(A_{LW}+\Delta A_{LW}\right)}{\partial \eta_{LW}}\cos\eta_{LW} - \frac{\partial H_1\left(A_{LW}+\Delta A_{LW}\right)}{\partial \eta_{LW}}\sin\eta_{LW}\Bigg\}\Bigg] \\
& + \frac{y_{cp}^{WP} k_L \omega_{LW}\left(\omega_{LW}+\sigma_{LW}\right)}{4A_{LW}}\Bigg[A_{LW}^2\left\{J_1(A_{LW})\cos\eta_{LW} + H_1(A_{LW})\sin\eta_{LW}\right\} \\
& + A_{LW}\frac{\partial \Delta A_{LW}}{\partial \eta_{LW}}\left\{J_1\left(A_{LW}+\Delta A_{LW}\right)\sin\eta_{LW} + H_1\left(A_{LW}+\Delta A_{LW}\right)\cos\eta_{LW}\right\} \\
& + A_{LW}\left(A_{LW}+\Delta A_{LW}\right)\Bigg\{J_1(A_{LW}+\Delta A_{LW})\cos\eta_{LW} \\
& - H_1(A_{LW}+\Delta A_{LW})\sin\eta_{LW} \\
& + \frac{\partial J_1\left(A_{LW}+\Delta A_{LW}\right)}{\partial \eta_{LW}}\sin\eta_{LW} + \frac{\partial H_1\left(A_{LW}+\Delta A_{LW}\right)}{\partial \eta_{LW}}\cos\eta_{LW}\Bigg\}\Bigg] \\
& + \frac{\Delta z_R^B k_L \omega_{LW}\left(\omega_{LW}+\sigma_{LW}\right)}{4}\left(A_{LW}+\Delta A_{LW}\right)\frac{\partial \Delta A_{LW}}{\partial \eta_{LW}}
\end{aligned} \tag{53}$$

$$
\begin{aligned}
\frac{\partial \overline{M}_{y\,LW}^{B}}{\partial \eta_{LW-1}} = & \frac{\omega_{LW}\left(\omega_{LW}+\sigma_{LW}\right)}{4A_{LW}}\left(\cos\alpha x_{cp}^{WP}k_L+\left\{\sin\alpha x_{cp}^{WP}+\Delta x_R^B\right\}k_D\right) \\
& * \left[A_{LW}\frac{\partial \Delta A_{LW}}{\partial \eta_{LW-1}}\left\{J_1\left(A_{LW}+\Delta A_{LW}\right)\cos\eta_{LW}-H_1\left(A_{LW}+\Delta A_{LW}\right)\sin\eta_{LW}\right\}\right. \\
& +\ A_{LW}\left(A_{LW}+\Delta A_{LW}\right) \\
& \left. * \left\{\frac{\partial J_1\left(A_{LW}+\Delta A_{LW}\right)}{\partial \eta_{LW-1}}\cos\eta_{LW}-\frac{\partial H_1\left(A_{LW}+\Delta A_{LW}\right)}{\partial \eta_{LW-1}}\sin\eta_{LW}\right\}\right] \\
& +\frac{y_{cp}^{WP}k_L\omega_{LW}\left(\omega_{LW}+\sigma_{LW}\right)}{4A_{LW}}\Bigg[\\
& +A_{LW}\frac{\partial \Delta A_{LW}}{\partial \eta_{LW-1}}\left\{J_1\left(A_{LW}+\Delta A_{LW}\right)\sin\eta_{LW}+H_1\left(A_{LW}+\Delta A_{LW}\right)\cos\eta_{LW}\right\} \\
& +\ A_{LW}\left(A_{LW}+\Delta A_{LW}\right)\Bigg\{ \\
& +\ \frac{\partial J_1\left(A_{LW}+\Delta A_{LW}\right)}{\partial \eta_{LW-1}}\sin\eta_{LW}+\frac{\partial H_1\left(A_{LW}+\Delta A_{LW}\right)}{\partial \eta_{LW-1}}\cos\eta_{LW}\Bigg\}\Bigg] \\
& +\frac{\Delta z_R^B k_L\omega_{LW}\left(\omega_{LW}+\sigma_{LW}\right)}{4}\left(A_{LW}+\Delta A_{LW}\right)\frac{\partial \Delta A_{LW}}{\partial \eta_{LW-1}} \qquad (54)
\end{aligned}
$$

F. Yawing Moment Control Derivatives

$$
\begin{aligned}
\frac{\partial \overline{M}_{z\,RW}^{B}}{\partial \delta_{RW}} = & \frac{-A_{RW}\omega_{RW}}{2}YM_{1_{RW}}+\frac{1}{4}\left[k_DA_{RW}YM_{2_{RW}}-k_LA_{RW}YM_{3_{RW}}\right. \\
& \left. +\ k_D\left(A_{RW}+\Delta A_{RW}\right)YM_{4_{RW}}-k_L\left(A_{RW}+\Delta A_{RW}\right)YM_{5_{RW}}\right] \\
& * \left(\frac{\omega_{RW}^3}{\left(\omega_{RW}-2\delta_{RW}\right)^2}\right) \qquad (55) \\
\frac{\partial \overline{M}_{z\,RW}^{B}}{\partial \omega_{RW}} = & \frac{A_{RW}\left(2\omega_{RW}-\delta_{RW}\right)}{2}YM_{1_{RW}}+\frac{1}{4}\left(k_DA_{RW}YM_{2_{RW}}-k_LA_{RW}YM_{3_{RW}}\right. \\
& \left. +\ k_D\left(A_{RW}+\Delta A_{RW}\right)YM_{4_{RW}}-k_L\left(A_{RW}+\Delta A_{RW}\right)YM_{5_{RW}}\right) \\
& * \left(\frac{2\omega_{RW}^3-7\delta_{RW}\omega_{RW}^2+4\omega_{RW}\delta_{RW}^2}{\left(\omega_{RW}-2\delta_{RW}\right)^2}\right) \qquad (56)
\end{aligned}
$$

where

$$\begin{aligned}
YM_{1_{RW}} &= J_1(A_{RW})\sin\eta_{RW}\left\{-k_D\left(\sin\alpha x_{cp}^{WP}+\Delta x_R^B\right)-\cos\alpha x_{cp}^{WP}k_L-y_{cp}^{WP}k_L\frac{\cos\eta_{RW}}{\sin\eta_{RW}}\right\} \\
&\quad -\frac{wk_LA_{RW}}{4} && (57) \\
YM_{2_{RW}} &= \left(\sin\alpha x_{cp}^{WP}+\Delta x_R^B\right)\left\{J_1(A_{RW})\sin\eta_{RW}-H_1(A_{RW})\cos\eta_{RW}\right\} && (58) \\
YM_{3_{RW}} &= -\cos\alpha x_{cp}^{WP}\left\{J_1(A_{RW})\sin\eta_{RW}-H_1(A_{RW})\cos\eta_{RW}\right\} \\
&\quad +y_{cp}^{WP}\left\{J_1(A_{RW})\cos\eta_{RW}+H_1(A_{RW})\sin\eta_{RW}\right\}+\frac{wA_{RW}}{4} && (59) \\
YM_{4_{RW}} &= \left(\sin\alpha x_{cp}^{WP}+\Delta x_R^B\right)\left\{J_1(A_{RW}+\Delta A_{RW})\sin\eta_{RW}\right. \\
&\quad \left.+\,H_1(A_{RW}+\Delta A_{RW})\cos\eta_{RW}\right\} && (60) \\
YM_{5_{RW}} &= -\cos\alpha x_{cp}^{WP}\left\{J_1(A_{RW}+\Delta A_{RW})\sin\eta_{RW}+H_1(A_{RW}+\Delta A_{RW})\cos\eta_{RW}\right\} \\
&\quad +y_{cp}^{WP}\left\{J_1(A_{RW}+\Delta A_{RW})\cos\eta_{RW}-H_1(A_{RW}+\Delta A_{RW})\sin\eta_{RW}\right\} \\
&\quad +\frac{w\left\{A_{RW}+\Delta A_{RW}\right\}}{4} && (61)
\end{aligned}$$

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$$
\begin{aligned}
\frac{\partial \overline{M}_{zRW}^{B}}{\partial \eta_{RW}} = & \frac{A_{RW}\omega_{RW}\left(\omega_{RW}-\delta_{RW}\right)}{2}\Bigg[J_1(A_{RW})\cos\eta\left\{-k_D\left(\sin\alpha x_{cp}^{WP}+\Delta x_R^B\right)\right. \\
& \left.-\ \cos\alpha x_{cp}^{WP}k_L\right\}+J_1(A_{RW})y_{cp}^{WP}k_L\sin\eta_{RW}\Bigg] \\
& +\frac{k_D A_{RW}\omega_{RW}\left(\omega_{RW}+\sigma_{RW}\right)}{4}\left(\sin\alpha x_{cp}^{WP}+\Delta x_R^B\right) \\
& *\left(J_1(A_{RW})\cos\eta_{RW}+H_1(A_{RW})\sin\eta_{RW}\right) \\
& -\frac{k_L A_{RW}\omega_{RW}\left(\omega_{RW}+\sigma_{RW}\right)}{4}\Bigg(-\cos\alpha x_{cp}^{WP}\left\{J_1(A_{RW})\cos\eta_{RW}+H_1(A_{RW})\sin\eta_{RW}\right\} \\
& +\ y_{cp}^{WP}\left\{-J_1(A_{RW})\sin\eta_{RW}+H_1(A_{RW})\cos\eta_{RW}\right\}\Bigg) \\
& +\frac{k_D\omega_{RW}\left(\omega_{RW}+\sigma_{RW}\right)\left(\sin\alpha x_{cp}^{WP}+\Delta x_R^B\right)}{4} \\
& *\Bigg[\frac{\partial \Delta A_{RW}}{\partial \eta_{RW}}\left\{J_1(A_{RW}+\Delta A_{RW})\sin\eta_{RW}+H_1(A_{RW}+\Delta A_{RW})\cos\eta_{RW}\right\} \\
& +\left(A_{RW}+\Delta A_{RW}\right)\Bigg\{\frac{\partial J_1(A_{RW}+\Delta A_{RW})}{\partial \eta_{RW}}\sin\eta_{RW}+J_1(A_{RW}+\Delta A_{RW})\cos\eta_{RW} \\
& +\ \frac{\partial H_1(A_{RW}+\Delta A_{RW})}{\partial \eta_{RW}}\cos\eta_{RW}-H_1(A_{RW}+\Delta A_{RW})\sin\eta_{RW}\Bigg\}\Bigg] \\
& -\frac{k_L\omega_{RW}(\omega_{RW}+\sigma_{RW})}{4}\Bigg[\frac{\partial \Delta A_{RW}}{\partial \eta_{RW}}\Bigg\{-\cos\alpha x_{cp}^{WP}\left(J_1(A_{RW}+\Delta A_{RW})\sin\eta_{RW}\right. \\
& \left.+\ H_1(A_{RW}+\Delta A_{RW})\cos\eta_{RW}\right) \\
& +\ y_{cp}^{WP}\Bigg\{J_1(A_{RW}+\Delta A_{RW})\cos\eta_{RW}-H_1(A_{RW}+\Delta A_{RW})\sin\eta_{RW}\Bigg\} \\
& +\ \frac{w(A_{RW}+\Delta A_{RW})}{4}\Bigg\} \\
& +\left(A_{RW}+\Delta A_{RW}\right)\Bigg\{-\cos\alpha x_{cp}^{WP}\Bigg(\frac{\partial J_1(A_{RW}+\Delta A_{RW})}{\partial \eta_{RW}}\sin\eta_{RW} \\
& +\ J_1(A_{RW}+\Delta A_{RW})\cos\eta_{RW}+\frac{\partial H_1(A_{RW}+\Delta A_{RW})}{\partial \eta_{RW}}\cos\eta_{RW} \\
& -\ H_1(A_{RW}+\Delta A_{RW})\sin\eta_{RW}\Bigg) \\
& +\ y_{cp}^{WP}\Bigg(\frac{\partial J_1(A_{RW}+\Delta A_{RW})}{\partial \eta_{RW}}\cos\eta_{RW}-J_1(A_{RW}+\Delta A_{RW})\sin\eta_{RW} \\
& -\ \frac{\partial H_1(A_{RW}+\Delta A_{RW})}{\partial \eta_{RW}}\sin\eta_{RW}-H_1(A_{RW}+\Delta A_{RW})\cos\eta_{RW}\Bigg)+\frac{w}{4}\frac{\partial \Delta A_{RW}}{\partial \eta_{RW}}\Bigg\}\Bigg]
\end{aligned} \tag{62}
$$

$$
\begin{aligned}
\frac{\partial \overline{M}_{z\,RW}^{B}}{\partial \eta_{RW-1}} =& \frac{k_D \omega_{RW}\left(\omega_{RW}+\sigma_{RW}\right)\left(\sin\alpha x_{cp}^{WP}+\Delta x_R^B\right)}{4} \\
&* \left[\frac{\partial \Delta A_{RW}}{\partial \eta_{RW-1}}\left\{J_1(A_{RW}+\Delta A_{RW})\sin\eta_{RW}+H_1(A_{RW}+\Delta A_{RW})\cos\eta_{RW}\right\}\right. \\
&+ (A_{RW}+\Delta A_{RW})\left\{\frac{\partial J_1(A_{RW}+\Delta A_{RW})}{\partial \eta_{RW-1}}\sin\eta_{RW}\right. \\
&+ \left.\left.\frac{\partial H_1(A_{RW}+\Delta A_{RW})}{\partial \eta_{RW-1}}\cos\eta_{RW}\right\}\right] \\
&- \frac{k_L \omega_{RW}(\omega_{RW}+\sigma_{RW})}{4}\left[\frac{\partial \Delta A_{RW}}{\partial \eta_{RW-1}}\left\{-\cos\alpha x_{cp}^{WP}\left(J_1(A_{RW}+\Delta A_{RW})\sin\eta_{RW}\right.\right.\right. \\
&+ \left. H_1(A_{RW}+\Delta A_{RW})\cos\eta_{RW}\right) \\
&+ y_{cp}^{WP}\left\{J_1(A_{RW}+\Delta A_{RW})\cos\eta_{RW}-H_1(A_{RW}+\Delta A_{RW})\sin\eta_{RW}\right\} \\
&+ \left.\frac{w(A_{RW}+\Delta A_{RW})}{4}\right\} \\
&+ (A_{RW}+\Delta A_{RW})\left\{-\cos\alpha x_{cp}^{WP}\left(\frac{\partial J_1(A_{RW}+\Delta A_{RW})}{\partial \eta_{RW-1}}\sin\eta_{RW}\right.\right. \\
&+ \left.\frac{\partial H_1(A_{RW}+\Delta A_{RW})}{\partial \eta_{RW-1}}\cos\eta_{RW}\right) \\
&+ y_{cp}^{WP}\left(\frac{\partial J_1(A_{RW}+\Delta A_{RW})}{\partial \eta_{RW-1}}\cos\eta_{RW}\right. \\
&- \left.\left.\left.\frac{\partial H_1(A_{RW}+\Delta A_{RW})}{\partial \eta_{RW-1}}\sin\eta_{RW}\right)+\frac{w}{4}\frac{\partial \Delta A_{RW}}{\partial \eta_{RW-1}}\right\}\right]
\end{aligned} \tag{63}
$$

For the left wing, the yawing moment control derivatives become

$$
\begin{aligned}
\frac{\partial \overline{M}_{z\,LW}^{B}}{\partial \delta_{LW}} =& \frac{A_{LW}\omega_{LW}}{2}YM_{1_{LW}} - \frac{1}{4}\left[k_D A_{LW} YM_{2_{LW}} - k_L A_{LW} YM_{3_{LW}}\right. \\
&+ \left. k_D\left(A_{LW}+\Delta A_{LW}\right)YM_{4_{LW}} - k_L\left(A_{LW}+\Delta A_{LW}\right)YM_{5_{LW}}\right] \\
&* \left(\frac{\omega_{LW}^3}{\left(\omega_{LW}-2\delta_{LW}\right)^2}\right)
\end{aligned} \tag{64}
$$

$$
\begin{aligned}
\frac{\partial \overline{M}_{z\,LW}^{B}}{\partial \omega_{LW}} =& \frac{-A_{LW}\left(2\omega_{LW}-\delta_{LW}\right)}{2}YM_{1_{LW}} - \frac{1}{4}\left[k_D A_{LW} YM_{2_{LW}} - k_L A_{LW} YM_{3_{LW}}\right. \\
&+ \left. k_D\left(A_{LW}+\Delta A_{LW}\right)YM_{4_{LW}} - k_L\left(A_{LW}+\Delta A_{LW}\right)YM_{5_{LW}}\right] \\
&* \left(\frac{2\omega_{LW}^3 - 7\delta_{LW}\omega_{LW}^2 + 4\omega_{LW}\delta_{LW}^2}{\left(\omega_{LW}-2\delta_{LW}\right)^2}\right)
\end{aligned} \tag{65}
$$

where

$$
\begin{aligned}
YM_{1_{LW}} &= J_1(A_{LW}) \sin \eta_{LW} \left\{ -k_D \left(\sin \alpha x_{cp}^{WP} + \Delta x_L^B \right) - \cos \alpha x_{cp}^{WP} k_L - y_{cp}^{WP} k_L \frac{\cos \eta_{LW}}{\sin \eta_{LW}} \right\} \\
&\quad - \frac{w k_L A_{LW}}{4} && (66) \\
YM_{2_{LW}} &= \left(\sin \alpha x_{cp}^{WP} + \Delta x_L^B \right) \left\{ J_1(A_{LW}) \sin \eta_{LW} - H_1(A_{LW}) \cos \eta_{LW} \right\} && (67) \\
YM_{3_{LW}} &= - \cos \alpha x_{cp}^{WP} \left\{ J_1(A_{LW}) \sin \eta_{LW} - H_1(A_{LW}) \cos \eta_{LW} \right\} \\
&\quad + y_{cp}^{WP} \left\{ J_1(A_{LW}) \cos \eta_{LW} + H_1(A_{LW}) \sin \eta_{LW} \right\} + \frac{w A_{LW}}{4} && (68) \\
YM_{4_{LW}} &= \left(\sin \alpha x_{cp}^{WP} + \Delta x_L^B \right) \left\{ J_1(A_{LW} + \Delta A_{LW}) \sin \eta_{LW} \right. \\
&\quad \left. + H_1(A_{LW} + \Delta A_{LW}) \cos \eta_{LW} \right\} && (69) \\
YM_{5_{LW}} &= - \cos \alpha x_{cp}^{WP} \left\{ J_1(A_{LW} + \Delta A_{LW}) \sin \eta_{LW} + H_1(A_{LW} + \Delta A_{LW}) \cos \eta_{LW} \right\} \\
&\quad + y_{cp}^{WP} \left\{ J_1(A_{LW} + \Delta A_{LW}) \cos \eta_{LW} - H_1(A_{LW} + \Delta A_{LW}) \sin \eta_{LW} \right\} \\
&\quad + \frac{w \left\{ A_{LW} + \Delta A_{LW} \right\}}{4} && (70)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \overline{M}_{zLW}^{B}}{\partial \eta_{LW}} = &-\frac{A_{LW}\omega_{LW}\left(\omega_{LW}-\delta_{LW}\right)}{2}\Bigg[J_1(A_{LW})\cos\eta\left\{-k_D\left(\sin\alpha x_{cp}^{WP}+\Delta x_L^B\right)\right.\\
&-\left.\cos\alpha x_{cp}^{WP}k_L\right\}+J_1(A_{LW})y_{cp}^{WP}k_L\sin\eta_{LW}\Bigg]\\
&-\frac{k_D A_{LW}\omega_{LW}\left(\omega_{LW}+\sigma_{LW}\right)}{4}\left(\sin\alpha x_{cp}^{WP}+\Delta x_L^B\right)\\
&*\Bigg(J_1(A_{LW})\cos\eta_{LW}+H_1(A_{LW})\sin\eta_{LW}\Bigg)\\
&+\frac{k_L A_{LW}\omega_{LW}\left(\omega_{LW}+\sigma_{LW}\right)}{4}\Bigg(-\cos\alpha x_{cp}^{WP}\left\{J_1(A_{LW})\cos\eta_{LW}+H_1(A_{LW})\sin\eta_{LW}\right\}\\
&+y_{cp}^{WP}\left\{-J_1(A_{LW})\sin\eta_{LW}+H_1(A_{LW})\cos\eta_{LW}\right\}\Bigg)\\
&-\frac{k_D\omega_{LW}\left(\omega_{LW}+\sigma_{LW}\right)\left(\sin\alpha x_{cp}^{WP}+\Delta x_L^B\right)}{4}\\
&*\Bigg[\frac{\partial\Delta A_{LW}}{\partial\eta_{LW}}\left\{J_1(A_{LW}+\Delta A_{LW})\sin\eta_{LW}+H_1(A_{LW}+\Delta A_{LW})\cos\eta_{LW}\right\}\\
&+\left(A_{LW}+\Delta A_{LW}\right)\Bigg\{\frac{\partial J_1(A_{LW}+\Delta A_{LW})}{\partial\eta_{LW}}\sin\eta_{LW}+J_1(A_{LW}+\Delta A_{LW})\cos\eta_{LW}\\
&+\frac{\partial H_1(A_{LW}+\Delta A_{LW})}{\partial\eta_{LW}}\cos\eta_{LW}-H_1(A_{LW}+\Delta A_{LW})\sin\eta_{LW}\Bigg\}\Bigg]\\
&+\frac{k_L\omega_{LW}(\omega_{LW}+\sigma_{LW})}{4}\Bigg[\frac{\partial\Delta A_{LW}}{\partial\eta_{LW}}\Bigg\{-\cos\alpha x_{cp}^{WP}\left(J_1(A_{LW}+\Delta A_{LW})\sin\eta_{LW}\right.\\
&+\left.H_1(A_{LW}+\Delta A_{LW})\cos\eta_{LW}\right)\\
&+y_{cp}^{WP}\Bigg\{J_1(A_{LW}+\Delta A_{LW})\cos\eta_{LW}-H_1(A_{LW}+\Delta A_{LW})\sin\eta_{LW}\Bigg\}\\
&+\frac{w(A_{LW}+\Delta A_{LW})}{4}\Bigg\}\\
&+\left(A_{LW}+\Delta A_{LW}\right)\Bigg\{-\cos\alpha x_{cp}^{WP}\Bigg(\frac{\partial J_1(A_{LW}+\Delta A_{LW})}{\partial\eta_{LW}}\sin\eta_{LW}\\
&+J_1(A_{LW}+\Delta A_{LW})\cos\eta_{LW}+\frac{\partial H_1(A_{LW}+\Delta A_{LW})}{\partial\eta_{LW}}\cos\eta_{LW}\\
&-H_1(A_{LW}+\Delta A_{LW})\sin\eta_{LW}\Bigg)\\
&+y_{cp}^{WP}\Bigg(\frac{\partial J_1(A_{LW}+\Delta A_{LW})}{\partial\eta_{LW}}\cos\eta_{LW}-J_1(A_{LW}+\Delta A_{LW})\sin\eta_{LW}\\
&-\frac{\partial H_1(A_{LW}+\Delta A_{LW})}{\partial\eta_{LW}}\sin\eta_{LW}-H_1(A_{LW}+\Delta A_{LW})\cos\eta_{LW}\Bigg)+\frac{w}{4}\frac{\partial\Delta A_{LW}}{\partial\eta_{LW}}\Bigg\}\Bigg]
\end{aligned}
\tag{71}
$$

$$
\begin{aligned}
\frac{\partial \overline{M}_{zLW}^{B}}{\partial \eta_{LW-1}} = &-\frac{k_D \omega_{LW}\left(\omega_{LW}+\sigma_{LW}\right)\left(\sin\alpha x_{cp}^{WP}+\Delta x_L^B\right)}{4} \\
&* \left[\frac{\partial \Delta A_{LW}}{\partial \eta_{LW-1}}\left\{J_1(A_{LW}+\Delta A_{LW})\sin\eta_{LW}+H_1(A_{LW}+\Delta A_{LW})\cos\eta_{LW}\right\}\right. \\
&+ (A_{LW}+\Delta A_{LW})\left\{\frac{\partial J_1(A_{LW}+\Delta A_{LW})}{\partial \eta_{LW-1}}\sin\eta_{LW}\right. \\
&+ \left.\left.\frac{\partial H_1(A_{LW}+\Delta A_{LW})}{\partial \eta_{LW-1}}\cos\eta_{LW}\right\}\right] \\
&+ \frac{k_L \omega_{LW}(\omega_{LW}+\sigma_{LW})}{4}\left[\frac{\partial \Delta A_{LW}}{\partial \eta_{LW-1}}\left\{-\cos\alpha x_{cp}^{WP}\left(J_1(A_{LW}+\Delta A_{LW})\sin\eta_{LW}\right.\right.\right. \\
&+ \left. H_1(A_{LW}+\Delta A_{LW})\cos\eta_{LW}\right) \\
&+ y_{cp}^{WP}\left\{J_1(A_{LW}+\Delta A_{LW})\cos\eta_{LW}-H_1(A_{LW}+\Delta A_{LW})\sin\eta_{LW}\right\} \\
&+ \left.\frac{w(A_{LW}+\Delta A_{LW})}{4}\right\} \\
&+ (A_{LW}+\Delta A_{LW})\left\{-\cos\alpha x_{cp}^{WP}\left(\frac{\partial J_1(A_{LW}+\Delta A_{LW})}{\partial \eta_{LW-1}}\sin\eta_{LW}\right.\right. \\
&+ \left.\frac{\partial H_1(A_{LW}+\Delta A_{LW})}{\partial \eta_{LW-1}}\cos\eta_{LW}\right) \\
&+ y_{cp}^{WP}\left(\frac{\partial J_1(A_{LW}+\Delta A_{LW})}{\partial \eta_{LW-1}}\cos\eta_{LW}\right. \\
&- \left.\left.\left.\frac{\partial H_1(A_{LW}+\Delta A_{LW})}{\partial \eta_{LW-1}}\sin\eta_{LW}\right)+\frac{w}{4}\frac{\partial \Delta A_{LW}}{\partial \eta_{LW-1}}\right\}\right]
\end{aligned}
\tag{72}
$$

III. Aerodynamic Control Derivatives about Hover

The control of this vehicle in the vicinity of hover is of considerable interest. Therefore, the control derivatives are evaluated at the hover condition where $\omega_{RW} = \omega_{LW} = \omega_o$, $\delta_{RW} = \delta_{LW} = 0$, $\eta_{RW} = \eta_{LW} = 0$, and $\Delta A_{RW} = \Delta A_{LW} = 0$. Additionally, it is assumed that the nominal center-of-gravity of the vehicle and wing root hinges are aligned such that $\Delta z_R^B = \Delta z_L^B = 0$. When the nominal center-of-gravity and wing root hinges are not aligned, a non-zero cycle-averaged pitching moment is produced which rotates the lift vector, translates the vehicle, and does not allow hover. Lastly, since $\Delta A = \eta - \eta_{-1}$, $\frac{\partial \Delta A}{\partial \eta} = 1$ and $\frac{\partial \Delta A}{\partial \eta_{-1}} = -1$.

A. X-Body Axis Force Control Derivatives About Hover

$$\left.\frac{\partial \overline{F}_{x\,RW}^{B}}{\partial \delta_{RW}}\right|_{hover} = 0 \tag{73}$$

$$\left.\frac{\partial \overline{F}_{x\,LW}^{B}}{\partial \delta_{LW}}\right|_{hover} = 0 \tag{74}$$

$$\left.\frac{\partial \overline{F}_{x\,RW}^{B}}{\partial \omega_{RW}}\right|_{hover} = k_L A_{RW}^2 \omega_o \tag{75}$$

$$\left.\frac{\partial \overline{F}_{x\,LW}^{B}}{\partial \omega_{LW}}\right|_{hover} = k_L A_{LW}^2 \omega_o \tag{76}$$

$$\left.\frac{\partial \overline{F}_{x\,RW}^{B}}{\partial \eta_{RW}}\right|_{hover} = \frac{k_L \omega_o^2}{4} A_{RW} \tag{77}$$

$$\left.\frac{\partial \overline{F}_{x\,LW}^{B}}{\partial \eta_{LW}}\right|_{hover} = \frac{k_L \omega_o^2}{4} A_{LW} \tag{78}$$

$$\left.\frac{\partial \overline{F}_{x\,RW}^{B}}{\partial \eta_{RW-1}}\right|_{hover} = -\frac{k_L \omega_o^2}{4} A_{RW} \tag{79}$$

$$\left.\frac{\partial \overline{F}_{x\,LW}^{B}}{\partial \eta_{LW-1}}\right|_{hover} = -\frac{k_L \omega_o^2}{4} A_{LW} \tag{80}$$

B. Y-Body Axis Force Control Derivatives About Hover

$$\left.\frac{\partial \overline{F}^{B}_{y\,RW}}{\partial \delta_{RW}}\right|_{hover} = 0 \tag{81}$$

$$\left.\frac{\partial \overline{F}^{B}_{y\,LW}}{\partial \delta_{LW}}\right|_{hover} = 0 \tag{82}$$

$$\left.\frac{\partial \overline{F}^{B}_{y\,RW}}{\partial \omega_{RW}}\right|_{hover} = 0 \tag{83}$$

$$\left.\frac{\partial \overline{F}^{B}_{y\,LW}}{\partial \omega_{LW}}\right|_{hover} = 0 \tag{84}$$

$$\left.\frac{\partial \overline{F}^{B}_{y\,RW}}{\partial \eta_{RW}}\right|_{hover} = \frac{k_D \omega_o^2}{4} H_1(A_{RW}) \tag{85}$$

$$\left.\frac{\partial \overline{F}^{B}_{y\,LW}}{\partial \eta_{LW}}\right|_{hover} = -\frac{k_D \omega_o^2}{4} H_1(A_{LW}) \tag{86}$$

$$\left.\frac{\partial \overline{F}^{B}_{y\,RW}}{\partial \eta_{RW-1}}\right|_{hover} = -\frac{k_D \omega_o^2}{4} H_1(A_{RW}) \tag{87}$$

$$\left.\frac{\partial \overline{F}^{B}_{y\,LW}}{\partial \eta_{LW-1}}\right|_{hover} = \frac{k_D \omega_o^2}{4} H_1(A_{LW}) \tag{88}$$

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C. Z-Body Axis Force Control Derivatives About Hover

$$\left.\frac{\partial \overline{F}_{z\,RW}^{B}}{\partial \delta_{RW}}\right|_{hover} = -k_D A_{RW} J_1(A_{RW})\omega_o \tag{89}$$

$$\left.\frac{\partial \overline{F}_{z\,LW}^{B}}{\partial \delta_{LW}}\right|_{hover} = -k_D A_{LW} J_1(A_{LW})\omega_o \tag{90}$$

$$\left.\frac{\partial \overline{F}_{z\,RW}^{B}}{\partial \omega_{RW}}\right|_{hover} = 0 \tag{91}$$

$$\left.\frac{\partial \overline{F}_{z\,LW}^{B}}{\partial \omega_{LW}}\right|_{hover} = 0 \tag{92}$$

$$\left.\frac{\partial \overline{F}_{z\,RW}^{B}}{\partial \eta_{RW}}\right|_{hover} = \frac{-k_D J_1(A_{RW})\omega_o^2}{4} \tag{93}$$

$$\left.\frac{\partial \overline{F}_{z\,LW}^{B}}{\partial \eta_{LW}}\right|_{hover} = \frac{-k_D J_1(A_{LW})\omega_o^2}{4} \tag{94}$$

$$\left.\frac{\partial \overline{F}_{z\,RW}^{B}}{\partial \eta_{RW-1}}\right|_{hover} = \frac{k_D J_1(A_{RW})\omega_o^2}{4} \tag{95}$$

$$\left.\frac{\partial \overline{F}_{z\,LW}^{B}}{\partial \eta_{LW-1}}\right|_{hover} = \frac{k_D J_1(A_{LW})\omega_o^2}{4} \tag{96}$$

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D. Rolling Moment Control Derivatives About Hover

$$\left.\frac{\partial \overline{M}_{x\,RW}^{B}}{\partial \delta_{RW}}\right|_{hover} = -\frac{k_D A_{RW} \omega_o}{2}\left[y_{cp}^{WP} A_{RW} + w J_1(A_{RW})\right] \tag{97}$$

$$\left.\frac{\partial \overline{M}_{x\,LW}^{B}}{\partial \delta_{LW}}\right|_{hover} = \frac{k_D A_{LW} \omega_o}{2}\left[y_{cp}^{WP} A_{LW} + w J_1(A_{LW})\right] \tag{98}$$

$$\left.\frac{\partial \overline{M}_{x\,RW}^{B}}{\partial \omega_{RW}}\right|_{hover} = 0 \tag{99}$$

$$\left.\frac{\partial \overline{M}_{x\,LW}^{B}}{\partial \omega_{LW}}\right|_{hover} = 0 \tag{100}$$

$$\left.\frac{\partial \overline{M}_{x\,RW}^{B}}{\partial \eta_{RW}}\right|_{hover} = -\frac{k_D \omega_o^2}{8}\left[2y_{cp}^{WP} A_{RW} + w J_1(A_{RW})\right] \tag{101}$$

$$\left.\frac{\partial \overline{M}_{x\,LW}^{B}}{\partial \eta_{LW}}\right|_{hover} = \frac{k_D \omega_o^2}{8}\left[2y_{cp}^{WP} A_{LW} + w J_1(A_{LW})\right] \tag{102}$$

$$\left.\frac{\partial \overline{M}_{x\,RW}^{B}}{\partial \eta_{RW-1}}\right|_{hover} = \frac{k_D \omega_o^2}{8}\left[2y_{cp}^{WP} A_{RW} + w J_1(A_{RW})\right] \tag{103}$$

$$\left.\frac{\partial \overline{M}_{x\,LW}^{B}}{\partial \eta_{LW-1}}\right|_{hover} = -\frac{k_D \omega_o^2}{8}\left[2y_{cp}^{WP} A_{LW} + w J_1(A_{LW})\right] \tag{104}$$

E. Pitching Moment Control Derivatives About Hover

$$\left.\frac{\partial \overline{M}_{y\,RW}^{B}}{\partial \delta_{RW}}\right|_{hover} = A_{RW} J_1(A_{RW})\omega_o \left(\cos\alpha x_{cp}^{WP} k_L + \left\{\sin\alpha x_{cp}^{WP} + \Delta x_R^B\right\} k_D\right) \tag{105}$$

$$\left.\frac{\partial \overline{M}_{y\,LW}^{B}}{\partial \delta_{LW}}\right|_{hover} = A_{LW} J_1(A_{LW})\omega_o \left(\cos\alpha x_{cp}^{WP} k_L + \left\{\sin\alpha x_{cp}^{WP} + \Delta x_L^B\right\} k_D\right) \tag{106}$$

$$\left.\frac{\partial \overline{M}_{y\,RW}^{B}}{\partial \omega_{RW}}\right|_{hover} = 0 \tag{107}$$

$$\left.\frac{\partial \overline{M}_{y\,LW}^{B}}{\partial \omega_{LW}}\right|_{hover} = 0 \tag{108}$$

$$\left.\frac{\partial \overline{M}_{y\,RW}^{B}}{\partial \eta_{RW}}\right|_{hover} = A_{RW}\omega_o^2 y_{cp}^{WP} k_L J_1(A_{RW}) + \frac{\omega_o^2}{4} \tag{109}$$

$$* \left[\left(\cos\alpha x_{cp}^{WP} k_L + \left\{\sin\alpha x_{cp}^{WP} + \Delta x_R^B\right\} k_D \right) J_1(A_{RW}) \right. \tag{110}$$

$$\left. + y_{cp}^{WP} k_L H_1(A_{RW}) \right] \tag{111}$$

$$\left.\frac{\partial \overline{M}_{y\,LW}^{B}}{\partial \eta_{LW}}\right|_{hover} = A_{LW}\omega_o^2 y_{cp}^{WP} k_L J_1(A_{LW}) + \frac{\omega_o^2}{4} \tag{112}$$

$$* \left[\left(\cos\alpha x_{cp}^{WP} k_L + \left\{\sin\alpha x_{cp}^{WP} + \Delta x_L^B\right\} k_D \right) J_1(A_{LW}) \right. \tag{113}$$

$$\left. + y_{cp}^{WP} k_L H_1(A_{LW}) \right] \tag{114}$$

$$\left.\frac{\partial \overline{M}_{y\,RW}^{B}}{\partial \eta_{RW-1}}\right|_{hover} = -\frac{\omega_o^2}{4} \tag{115}$$

$$* \left[\left(\cos\alpha x_{cp}^{WP} k_L + \left\{\sin\alpha x_{cp}^{WP} + \Delta x_R^B\right\} k_D \right) J_1(A_{RW}) \right. \tag{116}$$

$$\left. + y_{cp}^{WP} k_L H_1(A_{RW}) \right] \tag{117}$$

$$\left.\frac{\partial \overline{M}_{y\,LW}^{B}}{\partial \eta_{LW-1}}\right|_{hover} = -\frac{\omega_o^2}{4} \tag{118}$$

$$* \left[\left(\cos\alpha x_{cp}^{WP} k_L + \left\{\sin\alpha x_{cp}^{WP} + \Delta x_L^B\right\} k_D \right) J_1(A_{LW}) \right. \tag{119}$$

$$\left. + y_{cp}^{WP} k_L H_1(A_{LW}) \right] \tag{120}$$

F. Yawing Moment Control Derivatives About Hover

$$\left.\frac{\partial \overline{M}^B_{z\,RW}}{\partial \delta_{RW}}\right|_{hover} = 0 \tag{121}$$

$$\left.\frac{\partial \overline{M}^B_{z\,LW}}{\partial \delta_{LW}}\right|_{hover} = 0 \tag{122}$$

$$\left.\frac{\partial \overline{M}^B_{z\,RW}}{\partial \omega_{RW}}\right|_{hover} = -2k_L\omega_o\left[A_{RW}y^{WP}_{cp}J_1(A_{RW}) + \frac{A^2_{RW}w}{4}\right] \tag{123}$$

$$\left.\frac{\partial \overline{M}^B_{z\,LW}}{\partial \omega_{LW}}\right|_{hover} = 2k_L\omega_o\left[A_{LW}y^{WP}_{cp}J_1(A_{LW}) + \frac{A^2_{LW}w}{4}\right] \tag{124}$$

$$\left.\frac{\partial \overline{M}^B_{z\,RW}}{\partial \eta_{RW}}\right|_{hover} = \left[\frac{k_D\omega_o^2}{4}H_1(A_{RW})\left(\sin\alpha x^{WP}_{cp} + \Delta x^B_R\right)\right. \tag{125}$$

$$\left. -\frac{k_L\omega_o^2}{4}\left(-\cos\alpha x^{WP}_{cp}H_1(A_{RW}) + y^{WP}_{cp}J_1(A_{RW}) + \frac{wA_{RW}}{2}\right)\right] \tag{126}$$

$$\left.\frac{\partial \overline{M}^B_{z\,LW}}{\partial \eta_{LW}}\right|_{hover} = -\left[\frac{k_D\omega_o^2}{4}H_1(A_{LW})\left(\sin\alpha x^{WP}_{cp} + \Delta x^B_L\right)\right. \tag{127}$$

$$\left. -\frac{k_L\omega_o^2}{4}\left(-\cos\alpha x^{WP}_{cp}H_1(A_{LW}) + y^{WP}_{cp}J_1(A_{LW}) + \frac{wA_{LW}}{2}\right)\right] \tag{128}$$

$$\left.\frac{\partial \overline{M}^B_{z\,RW}}{\partial \eta_{RW-1}}\right|_{hover} = -\left[\frac{k_D\omega_o^2}{4}H_1(A_{RW})\left(\sin\alpha x^{WP}_{cp} + \Delta x^B_R\right)\right. \tag{129}$$

$$\left. -\frac{k_L\omega_o^2}{4}\left(-\cos\alpha x^{WP}_{cp}H_1(A_{RW}) + y^{WP}_{cp}J_1(A_{RW}) + \frac{wA_{RW}}{2}\right)\right] \tag{130}$$

$$\left.\frac{\partial \overline{M}^B_{z\,LW}}{\partial \eta_{LW-1}}\right|_{hover} = \left[\frac{k_D\omega_o^2}{4}H_1(A_{LW})\left(\sin\alpha x^{WP}_{cp} + \Delta x^B_L\right)\right. \tag{131}$$

$$\left. -\frac{k_L\omega_o^2}{4}\left(-\cos\alpha x^{WP}_{cp}H_1(A_{LW}) + y^{WP}_{cp}J_1(A_{LW}) + \frac{wA_{LW}}{2}\right)\right] \tag{132}$$

G. Control Effectiveness Matrix

Using the general form of Equation 2, writing this expression for all three forces and moments, and separating out the control and non-control parameters yields

$$\begin{bmatrix} \Delta\overline{F}^B_{x_{des}} \\ \Delta\overline{F}^B_{y_{des}} \\ \Delta\overline{F}^B_{z_{des}} \\ \Delta\overline{M}^B_{x_{des}} \\ \Delta\overline{M}^B_{y_{des}} \\ \Delta\overline{M}^B_{z_{des}} \end{bmatrix} = \mathbf{B_{A1}} \begin{bmatrix} \delta_{RW} \\ \delta_{LW} \\ \Delta\omega_{RW} \\ \Delta\omega_{LW} \\ \eta_{RW} \\ \eta_{LW} \end{bmatrix} + \mathbf{B_{A2}} \begin{bmatrix} -\eta_{RW-1} \\ -\eta_{LW-1} \end{bmatrix} \tag{133}$$

where $\mathbf{B_{A1}}$ is the control effectiveness matrix, that contains the aerodynamic control derivatives evaluated at hover, and the vector on the left side of Equation 133 is a desired set of forces and moments, which are generated by a cycle-averaged control law that will be discussed shortly, to achieve a desired maneuver. Since η_{RW-1}, η_{LW-1} are not control variables, $\mathbf{B_{A2}}$ is not considered a control effectiveness matrix. Instead, the terms containing η_{RW-1}, η_{LW-1} simply produce a set of forces and moments that are only persistent over a single wingbeat cycle. Hence, Equation 133 can be manipulated to give

$$\begin{bmatrix} \Delta\overline{F}^B_{x_{des}} \\ \Delta\overline{F}^B_{y_{des}} \\ \Delta\overline{F}^B_{z_{des}} \\ \Delta\overline{M}^B_{x_{des}} \\ \Delta\overline{M}^B_{y_{des}} \\ \Delta\overline{M}^B_{z_{des}} \end{bmatrix} - \mathbf{B_{A2}} \begin{bmatrix} -\eta_{RW-1} \\ -\eta_{LW-1} \end{bmatrix} = \mathbf{B_{A1}} \begin{bmatrix} \delta_{RW} \\ \delta_{LW} \\ \Delta\omega_{RW} \\ \Delta\omega_{LW} \\ \eta_{RW} \\ \eta_{LW} \end{bmatrix} \tag{134}$$

The matrices $\mathbf{B_{A1}}$ and $\mathbf{B_{A2}}$ are explicitly expressed as

$$\mathbf{B_{A1}} = \begin{bmatrix} 0 & 0 & \left.\frac{\partial \overline{F}^B_{x\,RW}}{\partial \omega_{RW}}\right|_{hover} & \left.\frac{\partial \overline{F}^B_{x\,LW}}{\partial \omega_{LW}}\right|_{hover} & \left.\frac{\partial \overline{F}^B_{x\,RW}}{\partial \eta_{RW}}\right|_{hover} & \left.\frac{\partial \overline{F}^B_{x\,LW}}{\partial \eta_{LW}}\right|_{hover} \\ 0 & 0 & 0 & 0 & \left.\frac{\partial \overline{F}^B_{y\,RW}}{\partial \eta_{RW}}\right|_{hover} & \left.\frac{\partial \overline{F}^B_{y\,LW}}{\partial \eta_{LW}}\right|_{hover} \\ \left.\frac{\partial \overline{F}^B_{z\,RW}}{\partial \delta_{RW}}\right|_{hover} & \left.\frac{\partial \overline{F}^B_{z\,LW}}{\partial \delta_{LW}}\right|_{hover} & 0 & 0 & \left.\frac{\partial \overline{F}^B_{z\,RW}}{\partial \eta_{RW}}\right|_{hover} & \left.\frac{\partial \overline{F}^B_{z\,LW}}{\partial \eta_{LW}}\right|_{hover} \\ \left.\frac{\partial \overline{M}^B_{x\,RW}}{\partial \delta_{RW}}\right|_{hover} & \left.\frac{\partial \overline{M}^B_{x\,LW}}{\partial \delta_{LW}}\right|_{hover} & 0 & 0 & \left.\frac{\partial \overline{M}^B_{x\,RW}}{\partial \eta_{RW}}\right|_{hover} & \left.\frac{\partial \overline{M}^B_{x\,LW}}{\partial \eta_{LW}}\right|_{hover} \\ \left.\frac{\partial \overline{M}^B_{y\,RW}}{\partial \delta_{RW}}\right|_{hover} & \left.\frac{\partial \overline{M}^B_{y\,LW}}{\partial \delta_{LW}}\right|_{hover} & 0 & 0 & \left.\frac{\partial \overline{M}^B_{y\,RW}}{\partial \eta_{RW}}\right|_{hover} & \left.\frac{\partial \overline{M}^B_{y\,LW}}{\partial \eta_{LW}}\right|_{hover} \\ 0 & 0 & \left.\frac{\partial \overline{M}^B_{z\,RW}}{\partial \omega_{RW}}\right|_{hover} & \left.\frac{\partial \overline{M}^B_{z\,LW}}{\partial \omega_{LW}}\right|_{hover} & \left.\frac{\partial \overline{M}^B_{z\,RW}}{\partial \eta_{RW}}\right|_{hover} & \left.\frac{\partial \overline{M}^B_{z\,LW}}{\partial \eta_{LW}}\right|_{hover} \end{bmatrix} \tag{135}$$

and

$$\mathbf{B_{A2}} = \begin{bmatrix} \left.\frac{\partial \overline{F}^B_{x\,RW}}{\partial \eta_{RW-1}}\right|_{hover} & \left.\frac{\partial \overline{F}^B_{x\,LW}}{\partial \eta_{LW-1}}\right|_{hover} \\ \left.\frac{\partial \overline{F}^B_{y\,RW}}{\partial \eta_{RW-1}}\right|_{hover} & \left.\frac{\partial \overline{F}^B_{y\,LW}}{\partial \eta_{LW-1}}\right|_{hover} \\ \left.\frac{\partial \overline{F}^B_{z\,RW}}{\partial \eta_{RW-1}}\right|_{hover} & \left.\frac{\partial \overline{F}^B_{z\,LW}}{\partial \eta_{LW-1}}\right|_{hover} \\ \left.\frac{\partial \overline{M}^B_{x\,RW}}{\partial \eta_{RW-1}}\right|_{hover} & \left.\frac{\partial \overline{M}^B_{x\,LW}}{\partial \eta_{LW-1}}\right|_{hover} \\ \left.\frac{\partial \overline{M}^B_{y\,RW}}{\partial \eta_{RW-1}}\right|_{hover} & \left.\frac{\partial \overline{M}^B_{y\,LW}}{\partial \eta_{LW-1}}\right|_{hover} \\ \left.\frac{\partial \overline{M}^B_{z\,RW}}{\partial \eta_{RW-1}}\right|_{hover} & \left.\frac{\partial \overline{M}^B_{z\,LW}}{\partial \eta_{LW-1}}\right|_{hover} \end{bmatrix} \tag{136}$$

From the structure of $\mathbf{B_{A1}}$, it is found that this matrix has rank six. Hence, split-cycle constant period frequency modulation with wing bias has provided a technique to control 6 degrees-of-freedom with only two physical actuators. Fundamental frequency $(\omega_{RW}, \omega_{LW})$ can be used to control the x-body axis force and yawing moment. Likewise, the split-cycle parameters $(\delta_{RW}, \delta_{LW})$ can be used to control rolling moment and either z-body axis force or pitching moment. The wing bias parameters' primary function is to replace the bob-weight[3,4] as a pitching moment control parameter. The control allocation objective is to find the control vector, $\begin{bmatrix} \delta_{RW} & \delta_{LW} & \Delta\omega_{RW} & \Delta\omega_{LW} & \eta_{RW} & \eta_{LW} \end{bmatrix}^T$, such that Equation 134 is satisfied. The desired forces and moments in Equation 134 are generated by a feedback structure designed to track a trajectory. The control allocation scheme is a pseudo-inverse

and the control vector is computed as

$$
\mathbf{B_{A1}}^{+}\left\{\begin{bmatrix}\Delta\overline{F}_{x_{des}}^{B}\\ \Delta\overline{F}_{y_{des}}^{B}\\ \Delta\overline{F}_{z_{des}}^{B}\\ \Delta\overline{M}_{x_{des}}^{B}\\ \Delta\overline{M}_{y_{des}}^{B}\\ \Delta\overline{M}_{z_{des}}^{B}\end{bmatrix}-\mathbf{B_{A2}}\begin{bmatrix}-\eta_{RW-1}\\ -\eta_{LW-1}\end{bmatrix}\right\}=\begin{bmatrix}\delta_{RW}\\ \delta_{LW}\\ \Delta\omega_{RW}\\ \Delta\omega_{LW}\\ \eta_{RW}\\ \eta_{LW}\end{bmatrix} \tag{137}
$$

where $\mathbf{B_{A1}}^{+}$ is the pseudo-inverse of $\mathbf{B_{A1}}$. The control values, in Equation 137, are held constant over a wingbeat cycle and are applied to the oscillator generating the wingbeat forcing function.

IV. Results

The modeling and control analysis is now utilized in a six degree-of-freedom (DOF) model of the flapping wing MAV. The control design approach is the same as that used by Oppenheimer, et. al.[4] Figure 1 shows the structure of the controller. Position or attitude errors generate desired accelerations (or forces and moments). The forcing functions are

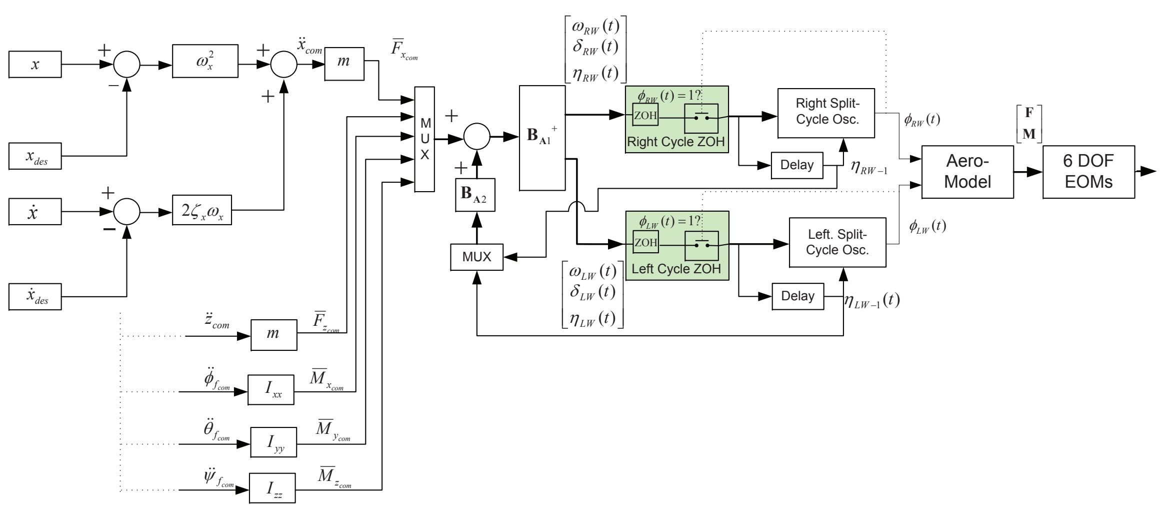

Figure 1. Five degree-of-freedom split-cycle controller with wing bias.

inputs to an aerodynamic model of the vehicle and 6 DOF equations of motion are integrated to obtain the state of the vehicle. The 6 DOF model is driven by the instantaneous forces and moments,[1] from the blade-element model, and not the cycle-averaged forces and moments.

The commands, for the presented simulation run, are shown in Figure 2. The vehicle

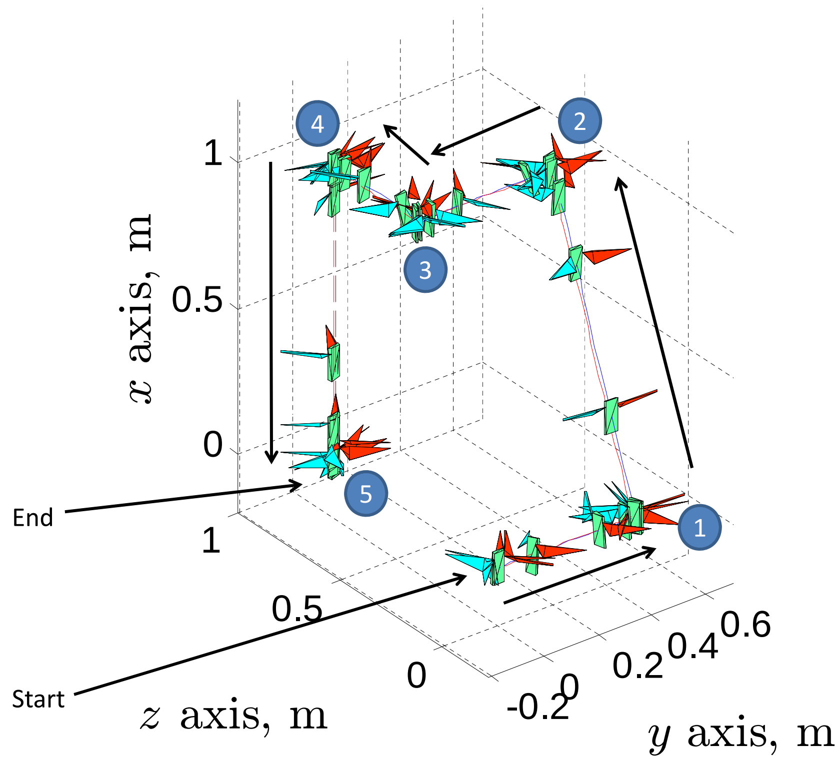

Figure 2. Commanded MAV Trajectory.

starts by executing a roll to align its heading with waypoint 1 and then pitches over to perform a translation in the direction of the y-axis of the inertial frame. Following this, the vehicle again aligns its heading with waypoint 2 and proceeds to translate in both the x-axis and z-axis directions of the inertial frame. Two constant altitude maneuvers are made followed by a change in altitude to get to the final waypoint (5). Figure 3 shows the roll, pitch, and yaw angles of the MAV during the flight, while Figure 4 shows the Euclidean norm of the position errors. The large roll angles are used to align the heading of the vehicle with the next waypoint, while, for forward translation, a non-zero pitch attitude angle is utilized. The position errors are small with a maximum of about 0.12 m.

Figures 5, 6, and 7 show the values of the control variables required to produce this maneuver. The trim wingbeat frequency is about 120 Hz with large excursions during the portions of the flight where altitude changes are made. The split cycle parameter is used to produce roll and is therefore active during all portions of the flight except movement to the last waypoint when there is no heading error. Similarly, the bias is used to pitch the vehicle to assist forward translation. Hence, the bias is active during all portions of the flight. One interesting point here is that a bias value of less than about 0.6° is sufficient to regulate the attitude.

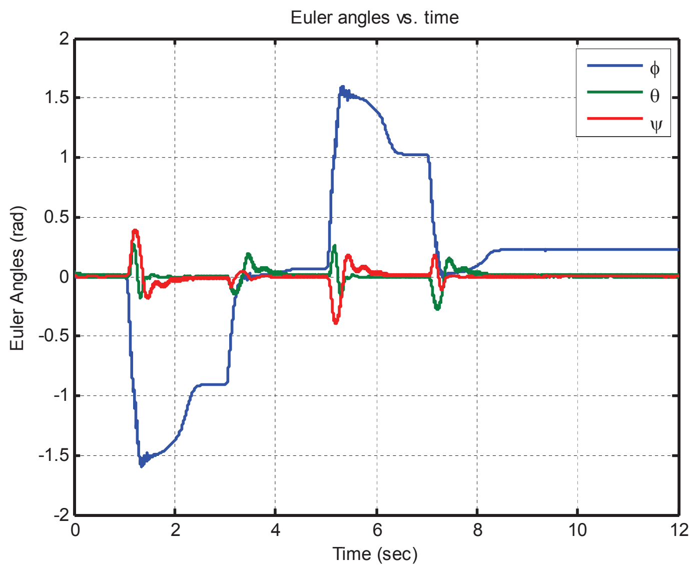

Figure 3. Vehicle Attitude.

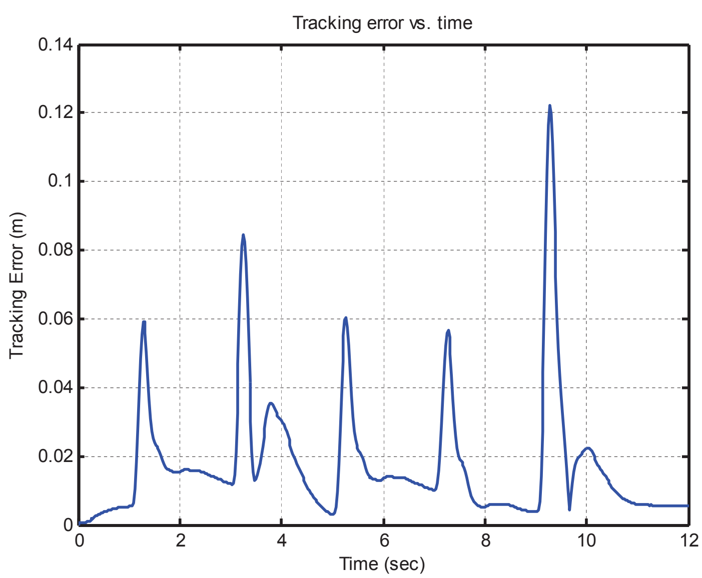

Figure 4. Euclidean Norm of Position Error.

V. Conclusions

In this work, a control strategy for a flapping wing micro air vehicle was developed. The control strategy made use of split-cycle constant period frequency modulation with wing bias to provide six DOF control over the vehicle. Simulation results show that sufficient control authority exists to track six DOF trajectories. The cycle-averaged controller provides the ability to track desired trajectories when applied to a blade-element based simulation model

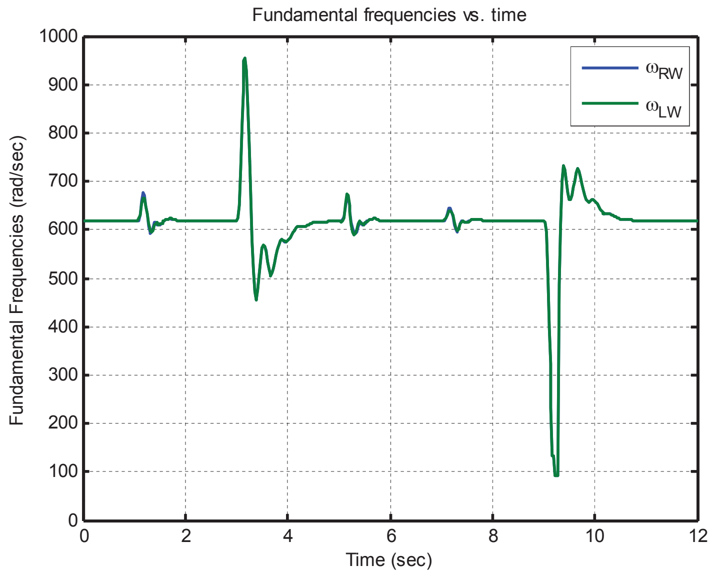

Figure 5. Fundamental Wingbeat Frequency, ω.

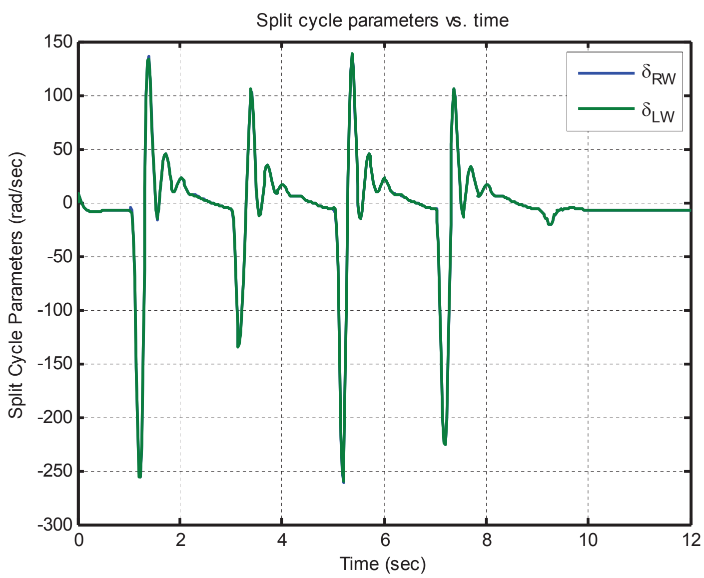

Figure 6. Split-Cycle Parameter, δ.

of the flapping wing MAV where the instantaneous aerodynamic forces and moments forced the equations of motion.

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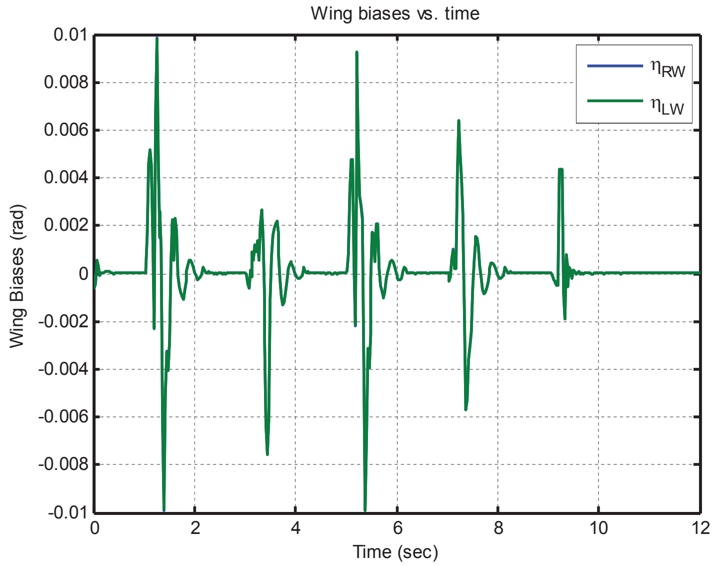

Figure 7. Wing Bias, η.

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